ON THE EFFECT OF PERIOD LENGTHS ON DYNAMIC STABILITY OF THIN BIPERIODIC CYLINDRICAL SHELLS

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ABSTRACT

The object of considerations is a thin linear-elastic cylindrical shell having a periodic structure (i.e. a periodically varying thickness and/or periodically varying elastic and inertial properties) in both directions tangent to the shell midsurface. Such shells are called biperiodic.

The aim of this paper is to propose a new averaged non-asymptotic model of biperiodic shells, which makes it possible to investigate parametric vibrations and dynamical stability of the shells under consideration.
As a tool of modeling we shall apply *the tolerance averaging technique*. The resulting equations have constant coefficients. Moreover, in contrast with models obtained by the known *asymptotic homogenization technique*, the proposed one takes into account the effect of the period lengths on the overall dynamic shell behavior, called *a length-scale effect*. It will be shown that this effect plays an important role in the dynamical stability analysis of the shells considered in this paper.

Key words: thin periodic cylindrical shell, dynamical stability, the length-scale effect

INTRODUCTION

In this paper, a new non-asymptotic model of dynamical stability analysis for thin cylindrical shells having a periodic structure (i.e. periodically varying thickness and/or periodically varying elastic and inertial properties) in both directions tangent to the shell midsurface M is presented. This situation is mainly oriented towards cylindrical shells reinforced by periodically spaced dense system of ribs as shown in Figure 1. Shells with a periodic structure along both directions tangent to M are termed *biperiodic*.

The periods of inhomogeneity are assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface. It means that the shells under consideration are composed of a large number of identical elements and every such element, called *a periodicity cell*, can be treated as a shallow shell.

The periodic cylindrical shells, being objects of considerations in this paper, are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as housings of reactors and tanks.

It should be noted that in the general case, on the shell midsurface we deal with not periodic but with what is called a locally periodic structure in directions tangent to M. Following [21], by a locally periodic shell we mean a shell which, in subregions of the shell midsurface M much smaller than M, can be approximately regarded as periodic. Hence, a locally periodic shell is made of a large number of not identical but similar elements. However, for cylindrical shells the Gaussian curvature is equal to zero and hence on the developable cylindrical surface we can separate a cell which can be referred to as the representative cell for a whole shell midsurface. It means that on cylindrical surface we deal with not a locally periodic but with periodic structure.

Because properties of periodic (or locally periodic) structures are described by highly oscillating, non-continuous, periodic functions, the exact equations of the shell (plate) theory are too complicated to apply to investigations of engineering problems. That is why a lot of different approximate modeling methods for shells and plates of this kind have been proposed. Periodic cylindrical shells and plates are usually described using *homogenized models* derived by means of *asymptotic methods*. These models represent certain equivalent structures with constant or slowly varying stiffnesses and averaged mass densities, cf. [5, 11, 12, 13, 14, 15]. Unfortunately, in models of this kind *the effect of the period lengths* (called also *the length-scale effect*) on the overall shell behavior is neglected.

The periodically densely ribbed plates and shells are also modeled as homogeneous orthotropic structures, cf. [1, 7]. These orthotropic models are not able to describe the length-scale effect on the overall shell (plate) behavior, being independent of the periods of inhomogeneity.

In order to analyze this effect, the *new averaged non-asymptotic models of thin uniperiodic cylindrical shells* (i.e. shells with a periodic structure along one direction tangent to M) have been proposed in [19] and [20]. These, co called, *the tolerance models* have been obtained by applying *the non-asymptotic tolerance averaging technique*, proposed and discussed for periodic composites in the monograph [22], to the known equations of Kirchhoff-Love-type cylindrical shells (differential equations with functional highly oscillating non-continuous periodic coefficients). These tolerance models have constant coefficients in periodicity direction and take into account the effect of a cell size on the global shell dynamics and stationary shell stability. This effect is described by means of certain extra unknowns called *internal or fluctuation variables* and by known functions which represent oscillations inside the periodicity cell, and are obtained either as approximate solutions to special eigenvalue problems for free vibrations on the separated cell with periodic boundary conditions or by using the finite element discretization of the cell. It has to be emphasized that the biperiodic shells, being subject-matter of considerations in this paper, are special cases of those with a periodic structure along one direction tangent to M and hence the models for uniperiodic shells can be applied to analyze the problems of biperiodic shells. However, the aforementioned *tolerance model of dynamic problems for periodic cylindrical shells* proposed in [19], and *that of stationary stability problems* given in [20] cannot be used to analyze a dynamical stability and parametric vibrations of the periodic shells. That is why, in this paper *the tolerance model of parametric vibration problems and dynamical stability problems for biperiodic Kirchhoff-Love-type cylindrical shells*, loaded by time-dependent forces tangent to the shell midsurface is derived and discussed.

It is worthy noting that the application of the tolerance averaging technique to the investigation of selected dynamic and stability problems for periodic plates can be found in many papers, e.g. in [17] and [3], where stability of densely stiffened Kirchhoff-type plates and of Hencky-Bolle-type plates is analyzed, respectively, in [8] and [16], where dynamics of Kirchhoff-type plates and of wavy-type plates is investigated, respectively. For more complete review of possible applications of the tolerance averaging technique to the modeling of micro-periodic structures the reader is referred to [22].

It has to be mentioned that an extremely extensive literature deals with elastic stability and dynamics of thin cylindrical shells reinforced by widely spaced stiffeners. Contrary to the shells with densely spaced ribs, which are objects of considerations in this paper, those having widely spaced stiffeners are analyzed with allowance for the discreteness in the arrangement of the ribs. It means that the dynamic and stability problems of such shells are considered within the framework of discrete models, while the dynamic and stability analysis of periodically, densely ribbed cylindrical shells investigated in this paper is carried out within continuum models. The discrete approach is in detail discussed in monographs [2, 6]. Moreover, in the mentioned monographs can be found an extensive review of papers and books dealing with stability and dynamic problems of widely ribbed shells as well as of densely stiffened shells treated as homogeneous orthotropic structures.

It is well known that stability problems of thin cylindrical shells being homogeneous or weakly heterogeneous have to be investigated by using the geometrically nonlinear shell theory, cf. [4, 10, 18]. However, in the case of the highly heterogeneous structures considered here (i.e. densely ribbed shells) which are described by using continuum

models, we are interested in the upper state of critical forces and hence we can use the geometrically linear stability theory for thin linear-elastic cylindrical Kirchhoff-Love type shells.

The aim of this contribution is three-fold :

- First, to formulate an averaged non-asymptotic model of thin biperiodic cylindrical shells which has constant coefficients and describes the effect of a cell size on the global dynamical stability of such shells. This model will be derived by using *the tolerance averaging procedure* proposed in [22].
- Second, to derive a simplified model (called asymptotic or homogenized) in which the length-scale effect is neglected.
- Third, to evaluate the effect of a cell size on the boundaries of two fundamental dynamic shell instability regions by using both the tolerance and homogenized models.

Basic denotations, preliminary concepts and starting equations will be presented in Section 2. The general line of the tolerance averaging approach will be shown in Section 3. The tolerance model for problems of dynamical stability of linear-elastic thin cylindrical shells with a periodic structure in two directions tangent to $\mathcal{M}$ will be proposed and discussed in Section 4. For comparison, the governing equations of a certain homogenized model will be given in Section 5. In the subsequent section, in order to evaluate the length-scale effect in dynamic stability problems, both the obtained tolerance and homogenized models will be applied to analyze the boundaries of two fundamental dynamic instability regions in closed circular cylindrical shell reinforced by longitudinal and circular ribs, which are densely and periodically distributed in circumferential and axial directions, respectively. Final remarks will be formulated in the last section.

PRELIMINARIES

In this paper we will investigate thin linear-elastic cylindrical shells with a periodic structure along both directions tangent to *M*. Cylindrical shells of this kind will be termed *biperiodic*. Example of such a shell is presented in Figure 1.

Figure 1. Example of biperiodic shell
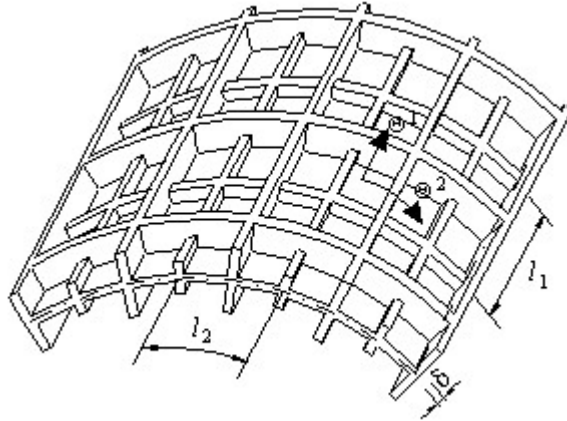
Denote by $\Omega \subset R^2$ a regular region of points $\mathbf{\Theta} \equiv (\Theta^1, \Theta^2)$ on the $O\Theta^1\Theta^2$-plane, Θ^1, Θ^2 being the Cartesian orthogonal coordinates on this plane and let E^3 be the physical space parametrized by the Cartesian orthogonal coordinate system $O\,x^1x^2x^3$. Let us introduce the orthogonal parametric representation of the undeformed smooth cylindrical shell midsurface M by means of : $M := \{\mathbf{x} \equiv (x^1, x^2, x^3) \in E^3 : \mathbf{x} = \mathbf{x}$ $(\Theta^1, \Theta^2), \mathbf{\Theta} \in \Omega \}$, where $\mathbf{x}(\Theta^1, \Theta^2)$ is a position vector of a point on M having coordinates Θ^1, Θ^2.

Throughout the paper indices α, β,... run over 1,2 and are related to the midsurface parameters Θ^1, Θ^2; indices A, B, ... run over 1,2,...,N, summation convention holds for all aforesaid indices.

The partial derivatives are indicated by a comma.

To every point $\mathbf{x}=\mathbf{x}(\boldsymbol{\Theta})$, $\boldsymbol{\Theta}\in\Omega$ we assign a covariant base vectors $\mathbf{a}_\alpha = \mathbf{x}_{,\alpha}$ and covariant midsurface first and second metric tensors denoted by $a_{\alpha\beta}, b_{\alpha\beta}$, respectively, which are given as follows : $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, $b_{\alpha\beta} = \mathbf{n}\cdot\mathbf{a}_{\alpha,\beta}$, where $\mathbf{n}$ is a unit vector normal to M.

Let $\delta(\boldsymbol{\Theta})$ stand for the shell thickness. We also define t as the time coordinate.

Taking into account that coordinate lines Θ^1=const and Θ^2=const are parallel on the $O\Theta^1\Theta^2$-plane and that Θ^1 and Θ^2 are arc coordinates on M (or axial and arc coordinates on M for a shell with periodic structure along a generatrix of the shell midsurface M and along the lines of principal curvature of M), we define l_1 and l_2 as the period lengths of the shell structure respectively in Θ^1- and Θ^2-directions on the $O\Theta^1\Theta^2$-plane. The period lengths l_1 and l_2 are assumed to be sufficiently large compared with the maximum shell thickness and sufficiently small as compared with the midsurface curvature radius R as well as the minimum characteristic length dimension L of the shell midsurface, i.e. $\delta_{max} << l_1, l_2 << min\{R,L\}$.

On the given above assumptions for the periods l_1 and l_2, the shell under consideration will be referred to as *a mezostructured shell*, cf. [21].

We shall denote by $\ddot{\text{A}} \equiv (-l_1/2, l_1/2)\times(-l_2/2, l_2/2)$ the plane element on the $O\Theta^1\Theta^2$-plane, which can be taken as a representative cell of the periodic shell structure (the periodicity cell). To every $\boldsymbol{\Theta}\in\Omega$ an arbitrary cell on $O\Theta^1\Theta^2$-plane will be defined by means of : $\ddot{\text{A}}(\boldsymbol{\Theta}) \equiv \boldsymbol{\Theta} + \ddot{\text{A}}, \boldsymbol{\Theta}\in\Omega_{\ddot{\text{A}}}$, $\Omega_{\ddot{\text{A}}} := \{\boldsymbol{\Theta}\in\Omega : \ddot{\text{A}}(\boldsymbol{\Theta})\subset\Omega\}$, where the point $\boldsymbol{\Theta}\in\Omega_{\ddot{\text{A}}}$ is a center of a cell $\ddot{\text{A}}(\boldsymbol{\Theta})$ and $\Omega_{\ddot{\text{A}}}$ is a set of all the cell centers which are inside Ω.

Let $l\equiv\sqrt{l_1^2 + l_2^2}$ be the diameter of the cell $\ddot{\text{A}}$. The parameter l has to satisfy the same assumptions as the period lengths l_1, l_2, i.e. $\delta_{max} << l << min\{R,L\}$, and hence it will be called *the mezostructure length parameter*.

A function $f(\boldsymbol{\Theta})$ defined on $\Omega_{\ddot{\text{A}}}$ will be called $\ddot{\text{A}}$-periodic if for arbitrary points (Θ^1,Θ^2), $(\Theta^1\pm l_1,\Theta^2)$, $(\Theta^1,\Theta^2\pm l_2)$, $(\Theta^1\pm l_1,\Theta^2\pm l_2)$ it satisfies the condition: $f(\Theta^1,\Theta^2)= f(\Theta^1\pm l_1,\Theta^2)= f(\Theta^1,\Theta^2\pm l_2)= f(\Theta^1\pm l_1,\Theta^2\pm l_2)$ in the whole domain of its definition and it is not constant.

It is assumed that the cylindrical shell thickness as well as its elastic and inertial properties are $\ddot{\text{A}}$-periodic functions of $\boldsymbol{\Theta}\equiv(\Theta^1,\Theta^2)$. Shells like that are called *biperiodic*.

The above periodic heterogeneities can be also interpreted as those caused by a periodically spaced dense system of ribs, as shown in Figure 1.

For an arbitrary integrable function $\varphi(\cdot)$ defined on Ω, following [22], we define *the averaging operation*, given by :

$$<\varphi>(\boldsymbol{\Theta}) \equiv \frac{1}{l_1 l_2}\int_{\Delta(\boldsymbol{\Theta})}\varphi(\Psi^1,\Psi^2)\,d\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}\equiv(\Psi^1,\Psi^2)\in\ddot{\text{A}}(\boldsymbol{\Theta}), \quad \boldsymbol{\Theta}\equiv(\Theta^1,\Theta^2)\in\Omega_{\ddot{\text{A}}}, \qquad (1)$$

For a periodic function φ in $\boldsymbol{\Theta}$, its averaged value obtained from (1) is constant.

The denotation $\cong$ is used for a tolerance relation and $\simeq$ denotes an approximation due to the truncation of the infinite series.

Our considerations will be based on the simplified linear Kirchhoff-Love second-order theory of thin elastic shells in which terms depending on the second metric tensor of M are neglected in the formulae for curvature changes. Below, we quote the general formulations of the theory under consideration.

The Kirchhoff-Love shell equations

Let $u_\alpha(\Theta,t)$, $w(\Theta,t)$ stand for the midsurface shell displacements in directions tangent and normal to M, respectively. We denote by $\varepsilon_{\alpha\beta}(\Theta,t)$, $\kappa_{\alpha\beta}(\Theta,t)$ the membrane and curvature strain tensors and by $n^{\alpha\beta}(\Theta,t)$, $m^{\alpha\beta}(\Theta,t)$ the stress resultants and stress couples, respectively. The elastic properties of the shell are described by *2D*-shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\Theta)$, $B^{\alpha\beta\gamma\delta}(\Theta)$. Let $\mu(\Theta)$ stand for the shell mass density per midsurface unit area. Let $f_\alpha(\Theta,t)$, $f(\Theta,t)$ be external force components per midsurface unit area, respectively tangent and normal to M. We denote by $\overline{N}^{\alpha\beta}(t)$ the time-dependent compressive membrane forces in the shell midsurface.

Functions $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$, $B^{\alpha\beta\gamma\delta}(\Theta)$ and $\delta(\Theta)$ are Δ-periodic functions of Θ.

The equations of a shell theory under consideration consist of:

– the strain-displacement equations

$$\varepsilon_{\gamma\delta} = u_{(\gamma,\delta)} - b_{\gamma\delta}\, w \quad , \qquad\qquad \kappa_{\gamma\delta} = -w_{,\gamma\delta} \quad , \tag{2}$$

– the stress-strain relations

$$n^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \quad , \qquad\qquad m^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \quad , \tag{3}$$

– the equations of motion

$$n^{\alpha\beta}_{,\alpha} - \mu\, a^{\alpha\beta}\ddot{u}_\alpha + f^\beta = 0 \quad , \qquad\qquad m^{\alpha\beta}_{,\alpha\beta} + b_{\alpha\beta}\, n^{\alpha\beta} - \overline{N}^{\alpha\beta} w_{,\alpha\beta} - \mu\, \ddot{w} + f = 0 \quad . \tag{4}$$

In the above equations the displacements $u_\alpha(\Theta,t)$ and $w(\Theta,t)$, $\Theta \in \Omega$, are the basic unknowns.

For biperiodic shells, $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$ and $B^{\alpha\beta\gamma\delta}(\Theta)$, $\Theta \in \Omega$, are non-continuous highly oscillating Ä -periodic functions; that is why equations (2)-(4) cannot be directly applied to the numerical analysis of special problems. From (2)-(4) an averaged non-asymptotic model of biperiodic cylindrical shells having coefficients, which are independent of Θ^1 - and Θ^2 -midsurface parameters as well as describing the cell size effect on the global shell behavior, will be derived. In order to derive it, *the tolerance averaging procedure* given in [22], will be applied. To make the analysis more clear, in the next section we shall outline the basic concepts and the main assumptions of this approach, following the monograph [22].

MODELING CONCEPTS AND ASSUMPTIONS

Following the monograph [22], we outline below the basic concepts and assumptions which will be used in the course of modeling procedure .

Basic concepts

The fundamental concepts of the tolerance averaging approach are those of a certain *tolerance system, slowly varying functions, periodic-like functions and periodic-like oscillating functions*. These functions will be defined with respect to the Ä -periodic shell structure defined in the foregoing section.
By *a tolerance system* we shall mean a pair $T=(\mathrm{F}, \mathcal{E}(\cdot))$, where F is a set of real-valued bounded functions $F(\cdot)$ defined on $\overline{\Omega}$ and their derivatives (including also time derivatives), which represent the unknowns in the problem under consideration (such as unknown shell displacements tangent and normal to M) and for which the tolerance

parameters ε_F being positive real numbers and determining the admissible accuracy related to computations of values of $F(\cdot)$ are given ; by ε is denoted the mapping $F\ni F(\cdot)\rightarrow \varepsilon_F$.

A continuous bounded differentiable function $F(\Theta,t)$ defined on $\overline{\Omega}$ is called $\ddot{A}$ -*slowly_varying* with respect to the cell $\ddot{A}$ and the tolerance system T, $F\in SV_{\ddot{A}}(T)$, if for every $\Theta,\Psi\in\overline{\Omega}$ such that $\|\Theta-\Psi\|\leq l$ the following condition holds $|F(\Theta)-F(\Psi)|\leq\varepsilon_F$. Roughly speaking, a function $F(\Theta,t)$ defined on $\overline{\Omega}$ is $\ddot{A}$ -*slowly_varying* if in the framework of tolerance can be treated (together with its derivatives) as constant on an arbitrary periodicity cell $\ddot{A}$.

The continuous function $\varphi(\cdot)$ defined on $\overline{\Omega}$ will be termed *a* $\ddot{A}$ -*periodic-like function*, $\varphi(\cdot)\in PL_{\ddot{A}}(T)$, with respect to the cell $\ddot{A}$ and the tolerance system T, if for every $\Theta=(\Theta^1,\Theta^2)\in\overline{\Omega}$ there exists a continuous $\ddot{A}$ -periodic function $\varphi_{\Theta}(\cdot)$ such that $(\forall\Psi=(\Psi^1,\Psi^2))\left\|\Theta-\Psi\right\|\leq l\Rightarrow\varphi(\Psi)\cong\varphi_{\Theta}(\Psi)\right]$, $\Psi\in\overline{\Omega}$, and the similar conditions are also fulfilled by all its derivatives. It means that the values of a periodic-like function $\varphi(\cdot)$ in an arbitrary cell $\ddot{A}(\Theta),\Theta\in\Omega_{\ddot{A}}$, can be approximated, with sufficient accuracy, by corresponding values of a certain $\ddot{A}$ -periodic function $\varphi_{\Theta}(\cdot)$. The function $\varphi_{\Theta}(\cdot)$ will be referred to as a $\ddot{A}$ -periodic approximation of $\varphi(\cdot)$ on $\ddot{A}(\Theta)$.

Let $\mu(\cdot)$ be a positive value $\ddot{A}$ -periodic function. *The periodic-like function* φ is called $\ddot{A}$ -*oscillating* (with the weight μ), $\varphi(\cdot)\in PL_{\ddot{A}}^{\mu}(T)$, provided that the condition $<\mu\varphi>(\Theta)\cong 0$ holds for every $\Theta\in\Omega_{\ddot{A}}.$ In the special case $\mu=$const *the oscillating periodic-like functions* satisfies condition $<\varphi>(\Theta)\cong 0$, $\Theta\in\Omega_{\ddot{A}}$; in this case we shall write $\varphi\in PL_{\Delta}^{1}(T)$.

In the subsequent considerations, the following propositions will be used:

(P1) If $\varphi(\cdot)\in PL_{\ddot{A}}(T)$ and f is bounded $\ddot{A}$ -periodic function then $<f\varphi>(\cdot)\in SV_{\ddot{A}}(T)$,

(P2) If $\varphi(\cdot)\in PL_{\ddot{A}}(T)$ then there exists the decomposition $\varphi(\cdot)=\varphi^0(\cdot)+\widetilde{\varphi}(\cdot)$, where $\varphi^0(\cdot)\in SV_{\ddot{A}}(T)$ and $\widetilde{\varphi}(\cdot)\in PL_{\Delta}^{\mu}(T)$, moreover, it can be shown that $\varphi^0(\cdot)\cong<\mu\varphi>(\cdot)<\mu>^{-1}$,

(P3) If $F\in SV_{\ddot{A}}(T)$ and f is bounded continuous $\ddot{A}$ -periodic function then $<fF>\in PL_{\ddot{A}}(T)$,

(P4) If $F\in SV_{\ddot{A}}(T)$, $G\in SV_{\ddot{A}}(T)$, $kF+mG\in F$ for some reals k,m, then $kF+mG\in SV_{\ddot{A}}(T)$.

The proofs of these propositions can be found in [22].

Modeling assumptions
The tolerance averaging technique is based on two modeling assumptions. The first of them is strictly related to the concept of $\ddot{A}$ -slowly varying and $\ddot{A}$ -periodic-like functions.

Tolerance Averaging Assumption. *If* $F\in SV_{\ddot{A}}(T)$, $\varphi(\cdot)\in PL_{\ddot{A}}(T)$ *and* $\varphi_{\Theta}(\cdot)$ *is a* $\ddot{A}$ -*periodic approximation of* $\varphi(\cdot)$ *on* $\ddot{A}(\Theta)$ *then for every* $\ddot{A}$ -*periodic bounded function* $f(\cdot)$ *and every continuous* $\ddot{A}$ -*periodic differentiable function* $h(\cdot)$ *such that* $\sup\{|h(\Psi^1,\Psi^2)|, (\Psi^1,\Psi^2)\in\ddot{A}\}\leq l$, *the following tolerance averaging relations determined by the pertinent tolerance parameters hold for every* $\Theta\in\Omega_{\ddot{A}}$:

(T1) $<fF>(\Theta)\cong<f>(\Theta)F(\Theta)$,   **(T2)** $<f(hF)_{,\alpha}>(\Theta)\cong<fFh_{,\alpha}>(\Theta)$,

(T3) $<f\varphi>(\Theta)\cong<f\varphi_{\Theta}>(\Theta)$,   **(T4)** $<h(f\varphi)_{,\alpha}>(\Theta)\cong-<f\varphi h_{,\alpha}>(\Theta)$.

It means that in the course of averaging the left-hand sides of formulae **(T1)-(T4)** can be approximated by their right-hand sides, respectively.

The second modeling assumption is based on heuristic premises.

Conformability Assumption. *In every periodic solid the displacement fields have to conform to the periodic structure of this solid. It means that the displacement fields are periodic-like functions and hence can be represented by a sum of averaged displacements, which are slowly varying (with respect to the cell and tolerance system), and by highly oscillating periodic-like disturbances, caused by the periodic structure of the solid.*

The aforementioned *Conformability Assumption* together with the *Tolerance Averaging Assumption* constitute the foundations of *the tolerance averaging technique*. Using this technique *the tolerance model of dynamical stability problems for biperiodic cylindrical shells* will be derived in the subsequent section.

THE TOLERANCE MODEL

Modeling procedure

Let us assume that there is a certain tolerance system T = (F, $\mathcal{E}$ ($\cdot$)), where the set F consists of the unknown shell displacements tangent and normal to *M* and their derivatives.

The tolerance averaging approach to Eqs. (2)-(4) will be realized in five steps.

Step 1. From the *Conformability Assumption* and **(P2)**, it follows that the unknown shell displacements $u_\alpha(\Theta,t)$, $w(\Theta,t)$ in Eqs.(2)-(4) have to satisfy the conditions: $u_\alpha(\Theta,t) \in PL_\Delta(T)$, $w(\Theta,t) \in PL_\Delta(T)$ and hence can be decomposed into

$$u_\alpha(\Theta,t) = U_\alpha(\Theta,t) + d_\alpha(\Theta,t) \quad , \qquad w(\Theta,t) = W(\Theta,t) + p(\Theta,t) \quad , \tag{5}$$

where $U_\alpha(\Theta,t), W(\Theta,t) \in SV_\Delta(T)$ are the averaged parts of displacements $u_\alpha(\Theta,t)$, $w(\Theta,t)$, respectively, called *macrodisplacements* and defined by $U_\alpha(\cdot,t) \equiv <\mu>^{-1} <\mu u_\alpha>(\cdot,t)$, $W(\cdot,t) \equiv <\mu>^{-1} <\mu w>(\cdot,t)$, and $d_\alpha(\cdot,t), p(\cdot,t) \in PL_\Delta^\mu(T)$ are the fluctuating parts of displacements $u_\alpha(\Theta,t)$, $w(\Theta,t)$, respectively, such that $<\mu d_\alpha(\Theta,t)> = <\mu p(\Theta,t)> = 0$.

In the subsequent considerations, we will neglect the effect of the fluctuations $p(\Theta,t)$ on the shell stability (this effect will be studied in a separated paper). It means that in equation (4)$_2$ we shall approximate the term $\overline{N}^{\alpha\beta} w_{,\alpha\beta}$ by $\overline{N}^{\alpha\beta} W_{,\alpha\beta}$.

Step 2. Substituting the right-hand side of (5) into (4) and after the tolerance averaging of the resulting equations, we arrive at the equations

$$[< D^{\alpha\beta\gamma\delta}>(U_{\gamma,\delta} - b_{\gamma\delta}W) + < D^{\alpha\beta\gamma\delta} d_{\gamma,\delta}>(\Theta,t) +$$

$$-b_{\gamma\delta} < D^{\alpha\beta\gamma\delta} p >(\Theta,t)]_{,\alpha} - <\mu> a^{\alpha\beta}\ddot{U}_\alpha = - < f^\beta >(\Theta,t),$$

$$\tag{6}$$

$$[< B^{\alpha\beta\gamma\delta}>W_{,\gamma\delta} + < B^{\alpha\beta\gamma\delta} p_{,\gamma\delta}>(\Theta,t)]_{,\alpha\beta} - b_{\alpha\beta}[< D^{\alpha\beta\gamma\delta}>(\Theta^1)(U_{\gamma,\delta} - b_{\gamma\delta}W) +$$

$$+ < D^{\alpha\beta\gamma\delta} d_{\gamma,\delta}>(\Theta,t) - b_{\gamma\delta} < D^{\alpha\beta\gamma\delta} p >] + \overline{N}^{\alpha\beta}W_{,\alpha\beta} + <\mu>\ddot{W} = < f >(\Theta,t),$$

which must hold for every $\Theta \in \Omega_{\ddot{A}}$ and every time *t*.

The above averaging implies the condition $< f^\beta > (\Theta, t), < f > (\Theta, t) \in SV_{\ddot{A}}(T)$. This situation takes place if the shell external loadings satisfy the condition: $f^\beta(\Theta, t), f(\Theta, t) \in PL_\Delta(T)$. Subsequently we shall use the decomposition:

$$f^\beta(\cdot, t) = f_0^\beta(\cdot, t) + \widetilde{f}^\beta(\cdot, t), \quad f(\cdot, t) = f_0(\cdot, t) + \widetilde{f}(\cdot, t), \qquad \text{where} \qquad f_0^\beta(\cdot, t), f_0(\cdot, t) \in SV_\Delta(T),$$

$$\widetilde{f}^\beta(\cdot, t), \widetilde{f}(\cdot, t) \in PL_\Delta^1(T) \text{ and } < \widetilde{f}^\beta > (\Theta, t) = < \widetilde{f} > (\Theta, t) = 0.$$

Step 3. Multiplying Eqs.(4)$_1$ and (4)$_2$ by arbitrary Δ-periodic test functions d^*, p^*, respectively, such that $< \mu d^* > = < \mu p^* > = 0$, integrating these equations over $\ddot{A}(\Theta), \Theta \in \Omega_{\ddot{A}}$, and using the *Tolerance Averaging Assumption*, as well as denoting by $\widetilde{d}_\alpha, \widetilde{p}$ the $\ddot{A}$-periodic approximations of d_α, p, respectively, on $\ddot{A}(\Theta)$, we obtain the periodic problem on $\ddot{A}(\Theta)$ for functions $\widetilde{d}_\alpha(\Psi^1, \Psi^2, t)$, $\widetilde{p}(\Psi^1, \Psi^2, t)$, $(\Psi^1, \Psi^2) \in \ddot{A}(\Theta) = \ddot{A}(\Theta^1, \Theta^2)$, given by the following variational conditions

$$- < d_{,\alpha}^* D^{\alpha\beta\gamma\delta} \widetilde{d}_{\gamma,\delta} > + b_{\gamma\delta} < d_{,\alpha}^* D^{\alpha\beta\gamma\delta} \widetilde{p} > - < d^* \mu \ddot{\widetilde{d}} > a^{\alpha\beta} =$$
$$= - < d^* f^\beta > + < d_{,\alpha}^* D^{\alpha\beta\gamma\delta} > (U_{\gamma,\delta} - b_{\gamma\delta} W) \; ,$$

$$\tag{7}$$

$$< p_{,\alpha\beta}^* B^{\alpha\beta\gamma\delta} \widetilde{p}_{,\gamma\delta} > - b_{\alpha\beta} [< p^* D^{\alpha\beta\gamma\delta} \widetilde{d}_{\gamma,\delta} > - b_{\gamma\delta} < p^* D^{\alpha\beta\gamma\delta} \widetilde{p} >] + < p^* \mu \ddot{\widetilde{p}} > =$$
$$= < p^* f > + b_{\alpha\beta} < p^* D^{\alpha\beta\gamma\delta} > (U_{\gamma,\delta} - b_{\gamma\delta} W) - < p_{,\alpha\beta}^* B^{\alpha\beta\lambda\delta} > W_{,\gamma\delta} \; .$$

Conditions (7)$_1$ and (7)$_2$ must hold for every Δ-periodic test function d^* and for every Δ-periodic test function p^*, respectively.

Equations (6),(7) represent the basis for obtaining the tolerance model of thin linear elastic biperiodic cylindrical shells, which makes it possible to investigate free and forced vibrations, parametric vibrations, dynamical stability and stationary stability (after neglecting the inertial forces and time coordinate).

Step 4. In order to obtain solution to the periodic problem on $\ddot{A}(\Theta)$, given by the variational equations (7), we can apply the known orthogonalization method. Hence, for arbitrary $(\Psi^1, \Psi^2) \in \ddot{A}(\Theta)$, $\Theta = (\Theta^1, \Theta^2) \in \Omega_{\ddot{A}}$ we can look for solutions to the periodic problem (7) in the form of the finite series

$$\widetilde{d}_\alpha(\Psi^1, \Psi^2, t) = h^A(\Psi^1, \Psi^2) Q_\alpha^A(\Theta^1, \Theta^2, t),$$
$$\widetilde{p}(\Psi^1, \Psi^2, t) = g^A(\Psi^1, \Psi^2) V^A(\Theta^1, \Theta^2, t), \quad A = 1, 2, ..., N,$$

$$\tag{8}$$

in which the choice of a number N of terms in the finite sums determines different degrees of approximations and where $Q_\alpha^A(\Theta^1, \Theta^2, t), V^A(\Theta^1, \Theta^2, t)$ are new kinematic unknowns called *fluctuation variables*, being $\ddot{A}$-slowly varying functions, i.e. $Q_\alpha^A, V^A \in SV_\Delta(T)$. Moreover, $h^A(\Psi^1, \Psi^2)$, $g^A(\Psi^1, \Psi^2)$, *A=1,...,N,* are known in every problem under consideration, linear-independent, *l*-periodic functions such that $h^A, l h_{,2}^A, l^{-1} g^A, g_{,2}^A, l g_{,22}^A \in$ $5(l)$ and $\max | h^A(\Psi^1, \Psi^2) | \leq l$, $\max | g^A(\Psi^1, \Psi^2) | \leq l^2$ as well as $< \mu h^A > = < \mu g^A > = 0$ for every A and $< \mu h^A h^B > = = < \mu g^A g^B > = 0$ for every $A \neq B$.

Functions $h^A(\Psi^1, \Psi^2)$, $g^A(\Psi^1, \Psi^2)$, *A=1,2,...N,* in (8) can be derived from the periodic Finite Element Method discretization of the cell and hence will be referred to as *the shape functions*. It can be observed that in many cases this discretization of the cell requires a large number of finite elements and consequently the number *N*

of extra unknowns Q_α^A, V^A in (8) is also large. The functions $h^A(\Psi^1,\Psi^2)$, $g^A(\Psi^1,\Psi^2)$, *A=1,...,N,* can also be obtained as exact or approximate solutions to certain periodic eigenvalue problems on the cell describing free periodic vibrations of the cell. It means that the functions h^A, g^A represent the expected forms of free periodic vibration modes of an arbitrary cell and hence are referred to as the *mode-shape functions*. Following [19], this periodic eigenvalue problem of finding continuous $\ddot{A}$-periodic eigenfunctions $h_\alpha(\Psi^1,\Psi^2)$, $g(\Psi^1,\Psi^2)$, $(\Psi^1,\Psi^2) \in \ddot{A}(\Theta)$, $\Theta = (\Theta^1,\Theta^2) \in \Omega_{\ddot{A}}$ is given by the equations

$$[D^{\alpha\beta\gamma\delta}(\Psi^1,\Psi^2)h_{\gamma,\alpha}(\Psi^1,\Psi^2)]_{,\delta} + \mu(\Psi^1,\Psi^2)\omega^2 a^{\alpha\beta}h_\alpha(\Psi^1,\Psi^2) = 0,$$

$$(9)$$

$$[B^{\alpha\beta\gamma\delta}(\Psi^1,\Psi^2)g_{,\alpha\beta}(\Theta^1,\Psi^2)]_{,\gamma\delta} - \mu(\Psi^1,\Psi^2)\omega^2 g(\Psi^1,\Psi^2) = 0,$$

and by the periodic boundary conditions on the cell $\ddot{A}(\Theta)$ together with the continuity conditions inside $\ddot{A}(\Theta)$; by ω we have denoted the free vibration frequency. By averaging the above equations over $\ddot{A}(\Theta)$ we obtain $<\mu h_\alpha> = <\mu g> = 0$.

Thus, $[h_\alpha^1(\Psi^1,\Psi^2),g^1(\Psi^1,\Psi^2)],[h_\alpha^2(\Psi^1,\Psi^2),g^2(\Psi^1,\Psi^2)],\ldots$ is a sequence of eigenfunctions related to the sequence of eigenvalues $\left[\omega_\alpha^2,\omega^2\right]_1,\left[\omega_\alpha^2,\omega^2\right]_2,\ldots$. In the modeling procedure this sequence is restricted to the $N \geq 1$ eigenfunctions. Moreover, in most problems the analysis will be restricted to the simplest case *N=1* in which we take into account only the lowest natural vibration modes (in directions tangent and normal to *M*) related to Eqs. (9).

In this paper it is assumed that $h_1^A = h_2^A$ and hence we denote $h^A \equiv h_1^A = h_2^A$.

Step 5. Substituting the right-hand sides of (8) into (6) and (7) and setting $d^* = h^A(\Psi^1,\Psi^2)$, $p^* = g^A(\Psi^1,\Psi^2)$, *A=1,2,...,N,* in (7), on the basis of the *Tolerance Averaging Assumption* we arrive at the *tolerance model of dynamical stability problems for biperiodic cylindrical shells*. In the next subsection the equations of this model will be given and discussed.

Governing equations of the non-asymptotic model

In the previous subsection, applying the tolerance averaging of Kirchhoff-Love second-order shell equations we have arrived at the *tolerance model of dynamical stability problems for shells having a periodic structure along both directions tangent to the shell midsurface*.
Under extra denotations :

$$\widetilde{D}^{\alpha\beta\gamma\delta} \equiv <D^{\alpha\beta\gamma\delta}>, \quad D^{A\alpha\beta\gamma} \equiv <D^{\alpha\beta\gamma\delta}h_{,\delta}^A>,$$

$$L^{A\alpha\beta} \equiv l^{-2}b_{\gamma\delta}<D^{\alpha\beta\gamma\delta}g^A>, \quad \widetilde{B}^{\alpha\beta\gamma\delta} \equiv <B^{\alpha\beta\gamma\delta}>, \quad K^{A\alpha\beta} \equiv <B^{\alpha\beta\gamma\delta}g_{,\gamma\delta}^A>,$$

$$C^{AB\beta\gamma} \equiv <D^{\alpha\beta\gamma\delta}h_{,\alpha}^A h_{,\delta}^B>, \quad F^{AB\beta} \equiv l^{-2}b_{\gamma\delta}<D^{\alpha\beta\gamma\delta}h_{,\alpha}^A g^B>,$$

$$(10)$$

$$S^{AB} \equiv <B^{\alpha\beta\gamma\delta}g_{,\alpha\beta}^A g_{,\gamma\delta}^B>, \quad \overline{L}^{AB} \equiv l^{-4}b_{\alpha\beta}b_{\gamma\delta}<D^{\alpha\beta\gamma\delta}g^A g^B>,$$

$$\widetilde{\mu} \equiv <\mu>, \quad \widetilde{\mu}^{AB} \equiv l^{-2}<\mu h^A h^B>, \quad \overline{\mu}^{AB} \equiv l^{-4}<\mu g^A g^B>,$$

$$\widetilde{P}^{A\beta} \equiv l^{-1}<\widetilde{f}^\beta h^A>, \quad \widetilde{P}^A \equiv l^{-2}<\widetilde{f}g^A>,$$

this model is represented by :

– the constitutive equations

$$N^{\alpha\beta} = \widetilde{D}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + D^{B\alpha\beta\gamma}Q_\gamma^B - \underline{l^2 L^{B\alpha\beta}}\,V^B,$$

$$M^{\alpha\beta} = \widetilde{B}^{\alpha\beta\gamma\delta}W_{,\gamma\delta} + K^{B\alpha\beta}V^B,$$

(11)

$$H^{A\beta} = D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + C^{AB\beta\gamma}Q_\gamma^B - \underline{l^2 F^{AB\beta}}\,V^B,$$

$$G^A \equiv -\underline{l^2 L^{A\gamma\delta}}(U_{\gamma,\delta} - b_{\gamma\delta}W) + K^{A\alpha\beta}W_{,\alpha\beta} - \underline{l^2 F^{AB\gamma}}\,Q_\gamma^B + (S^{AB} + \underline{l^4 \overline{L}^{AB}})V^B,$$

– the system of three averaged partial differential equations of motion for macrodisplacements $U_\alpha(\Theta,t),\ W(\Theta,t)$

$$N^{\alpha\beta}_{,\alpha} - \widetilde{\mu}\,a^{\alpha\beta}\ddot{U}_\alpha + f_0^\beta = 0 \quad , \qquad M^{\alpha\beta}_{,\alpha\beta} - b_{\alpha\beta}N^{\alpha\beta} + \overline{N}^{\alpha\beta}W_{,\alpha\beta} + \widetilde{\mu}\,\ddot{W} - f_0 = 0 \quad , \qquad (12)$$

– the system of 3N ordinary differential equations for the fluctuation variables $Q_\alpha^B(\Theta,t),\ V^B(\Theta,t),\ B=1,2,...,N,$

$$\underline{l^2 \widetilde{\mu}^{AB}}\,a^{\gamma\beta}\ddot{Q}_\gamma^B + H^{A\beta} - \underline{l\widetilde{P}^{A\beta}} = 0 \quad ,$$

$$\underline{l^4 \overline{\mu}^{AB}}\ddot{V}^B + G^A - \underline{l^2 \widetilde{P}^A} = 0 , \qquad A,B = 1,2,...,N \quad ,$$

(13)

where some terms depend explicitly on the mezostructure length parameter *l*.

The above model has a physical sense provided that the basic unknowns $U_\alpha(\Theta,t), W(\Theta,t), Q_\gamma^A(\Theta,t), V^A(\Theta,t) \in SV_{\ddot{A}}(T),\ A=1,2,...,N,$ i.e. they are $\ddot{A}$ -slowly-varying functions of Θ^1 - and Θ^2 -midsurface parameters.

It can be observed that in the tolerance model equation (12) we deal with $\overline{N}^{\alpha\beta}(t) > 0$ if $\overline{N}^{\alpha\beta}(t)$ are compressive forces.

Taking into account (5) and (8) the shell displacement fields can be approximated by means of formulae

$$u_\alpha(\Theta,t) \simeq U_\alpha(\Theta,t) + h^A(\Psi^1,\Psi^2)Q_\alpha^A(\Theta,t),$$

$$w(\Theta,t) \simeq W(\Theta,t) + g^A(\Psi^1,\Psi^2)V^A(\Theta,t), \quad A = 1,2,...,N,$$

(14)

where the approximation $\simeq$ depends on the number of terms $h^A(\Psi^1,\Psi^2)Q_\alpha^A(\Theta,t),$ $g^A(\Psi^1,\Psi^2)V^A(\Theta,t)$.

The characteristic features of the derived model are:

- The model takes into account the effect of the cell size on the overall shell dynamics and stability; this effect is described by terms dependent explicitly on the mezostructure length parameter *l* .
- The model equations have constant coefficients.
- The number and form of boundary conditions for the macrodisplacements $U_\alpha(\Theta,t), W(\Theta,t)$ are the same as in the classical shell theory governed by equations (2)-(4). The fluctuation variables $Q_\gamma^A(\Theta,t), V^A(\Theta,t)$ are governed by the system of 3*N* ordinary differential equations involving only time derivatives; hence there are no extra boundary conditions for these functions, and that is why they play the role of kinematic internal variables.

- It is easy to see that in order to derive the governing equations (11)-(13), we have to previously obtained the periodic shape (mode-shape) functions $h^A(\Psi^1, \Psi^2)$, $g^A(\Psi^1, \Psi^2)$, $A=1,2,...,N$. These functions can be derived from the periodic finite element method discretization of the cell or obtained as solutions to the periodic eigenvalue problem describing free vibrations of the cell, given by (9). Moreover, it can be shown that the results obtained in the framework of the first approximation (i.e. for $N=1$) are sufficient from the computational point of view, provided that the functions $h \equiv h^1$, $g \equiv g^1$ are not derived from the periodic finite element method discretization of the cell but are taken as eigenfunctions related to the smallest eigenvalues of the eigenvalue cell problem given by means of equations (9).

For a homogeneous shell $\mu(\Theta)$, $D^{\alpha\beta\gamma\delta}(\Theta)$ and $B^{\alpha\beta\gamma\delta}(\Theta)$, $\Theta \in \Omega$, are constant and because $< \mu h^A > = < \mu g^A > =0$ we obtain $< h^A > = < g^A > =0$, and hence $< h^A_{,\alpha} > = < g^A_{,\alpha} > = = < g^A_{,\alpha\beta} > = 0$. In this case equations (12) reduce to the well known linear-elastic shell equations of motion for macrodisplacements $U_\alpha(\Theta,t), W(\Theta,t)$, and independently for $Q^A_\alpha(\Theta,t), V^A(\Theta,t)$ we arrive at a system of N differential equations. In the case under the condition $\widetilde{f}^\beta = \widetilde{f} = 0$ and for initial conditions $Q^A_\alpha(\Theta,t_0) = V^A(\Theta,t_0) = 0$, $A=1,2,...,N$, we obtain $Q^A_\alpha = V^A = 0$; hence the constitutive equations (11) and equations of motion (12) reduce to the starting equations (3) and (4), respectively.

It has to be emphasized that the tolerance model (11)-(13) has been derived in the framework of the geometrically linear stability theory for thin linear-elastic Kirchhoff-Love type shells. That is why, the model can be applied to analyze the problems of dynamic shell stability, provided that the upper critical forces are sufficient from the point of view of calculations made for solving those problems.

In the next section the homogenized model of biperiodic cylindrical shells under consideration will be derived as a special case of equations (11)-(13).

GOVERNING EQUATIONS OF THE ASYMPTOTIC MODEL

The simplified model of biperiodic cylindrical shells, called *homogenized* or *asymptotic*, can be derived directly from the tolerance model (11)-(13) by a limit passage $l \to 0$, i.e. by neglecting the terms which depend on the mezostructure length parameter l. Hence, Eqs.(13) yield :

$$C^{AB\beta\gamma}Q^B_\gamma = -D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W),$$
$$S^{AB}V^A = -K^{B\gamma\delta}W_{,\gamma\delta}. \tag{15}$$

From the positive definiteness of the strain energy it follows that $N \times N$ matrix of elements S^{AB} is non-singular, and the linear transformation determined by components $C^{AB\beta\gamma}$ is invertible. Hence a solution to equations (15) can be written in the form

$$Q^B_\gamma = -G^{BC}_{\gamma\eta} D^{C\eta\mu\vartheta}(U_{\mu,\vartheta} - b_{\mu\vartheta}W) \quad,$$
$$V^A = -E^{AB}K^{B\gamma\delta}W_{,\gamma\delta} \quad, \tag{16}$$

where $G^{AB}_{\alpha\beta}$ and E^{AB} are defined by $G^{AB}_{\alpha\beta}C^{BC\beta\gamma} = \delta^\gamma_\alpha \delta^{AC}$, $E^{AB}S^{BC} = \delta^{AC}$.
Setting

$$D^{\alpha\beta\gamma\delta}_{eff} \equiv \widetilde{D}^{\alpha\beta\gamma\delta} - D^{A\alpha\beta\eta}G^{AB}_{\eta\xi}D^{B\xi\gamma\delta}, \qquad B^{\alpha\beta\gamma\delta}_{eff} \equiv \widetilde{B}^{\alpha\beta\gamma\delta} - K^{A\alpha\beta}E^{AB}K^{B\gamma\delta} \quad,$$

and substituting the expression (16) into the constitutive equations (11)$_{1,2}$, in which the underlined term is neglected, we arrive at the homogenized (asymptotic) shell model governed by:

– *equations of motion*

$$D_{eff}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta\alpha} - b_{\gamma\delta}W_{,\alpha}) - <\mu> a^{\alpha\beta}\ddot{U}_\alpha + f_0^\beta = 0,$$
$$B_{eff}^{\alpha\beta\gamma\delta}W_{,\alpha\beta\gamma\delta} - b_{\alpha\beta}D_{eff}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + \overline{N}^{\alpha\beta}W_{,\alpha\beta} + <\mu>\ddot{W} - f_0 = 0,$$

(17)

– *constitutive equations*

$$N^{\alpha\beta} = D_{eff}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W), \qquad M^{\alpha\beta} = B_{eff}^{\alpha\beta\gamma\delta}W_{,\gamma\delta}$$

(18)

where $D_{eff}^{\alpha\beta\gamma\delta}$, $B_{eff}^{\alpha\beta\gamma\delta}$ are called the effective stiffnesses .

The obtained above homogenized model governed by Eqs.(17),(18) is not able to describe the length-scale effect on the overall shell behavior being independent of the cell size l .

In order to show differences between the results obtained from the tolerance biperiodic shell model (11)-(13), and from the homogenized model (17) and (18), the boundaries of two fundamental regions of the dynamic shell instability will be determined and analyzed in the next section.

APPLICATIONS

Now, the governing equations of both the models presented in the previous Sections will be applied to analyze the problem of dynamic stability of a closed circular cylindrical simply supported shell with L_1 as its axial length and with δ, R as its constant thickness and its midsurface curvature radius, respectively. The shell is reinforced by two families of longitudinal ribs (called stringers), which are periodically densely distributed in circumferential direction and, at the same time, the shell is reinforced by two families of circular ribs (called rings), which are periodically densely distributed along the axis of the shell; the fragment of such a shell is shown in Figure 2. The stiffeners of both families are assumed to have constant rectangular cross-sections with A_1, A_2 as their areas and with I_1, I_2 as their moments of inertia. Moreover, the gravity centers of the stiffener cross-sections are situated on the shell midsurface, cf. Figures 3 and 4. It is assumed that both the shell and stiffeners are made of homogeneous isotropic materials and let us denote by E, ν Young's modulus and Poisson's ratio of the shell material, respectively, and by E_1, E_2 Young's moduli of the rib materials. At the same time μ_0 stands for the constant shell mass density per midsurface unit area and μ_1, μ_2 stand for the constant mass densities of the stiffeners per the stiffener unit length, cf. Figures 3 and 4.

Let Θ^1, Θ^2 be axial and arc coordinates on the shell midsurface M, respectively, and let Θ^2-coordinate lines coincide with the lines of principal curvature of this surface.

It is assumed that the edges of the shell lie on the coordinate lines $\Theta^1 = 0, \Theta^1 = L_1$ and that all four edges are simply supported.

In agreement with considerations in Section 2., on $O\Theta^1\Theta^2$-plane we define l_1 and l_2 as the periods of the stiffened shell structure in Θ^1- and Θ^2-directions, respectively. In the subsequent considerations it will be assumed that the periods l_1 and l_2 are equal and hence let us introduce the denotation $l \equiv l_1 = l_2$. The period $l_1 = l$ represents the distance (i.e. the rectilinear length measured along the axial coordinate lines) between axes of two neighboring rings belonging to the same family, cf. Figures 2 and 3. The period $l_2 = l$ represents the distance (i.e. the arc length measured along the lines of midsurface principal curvature) between axes of two neighboring stringers belonging to the same family, cf. Figures 2 and 4. The period $l \equiv l_1 = l_2$ has to satisfy the conditions $\delta \ll l \ll \min(L_1, 2\pi R)$; it means that the number of stiffeners has to be very large.

Figure 2. A fragment of a shell with two families of biperiodically spaced ribs

$$l_2 << 2\pi R$$
$$l_1 << L_1$$
$$\delta << l_1$$
$$\delta << l_2$$
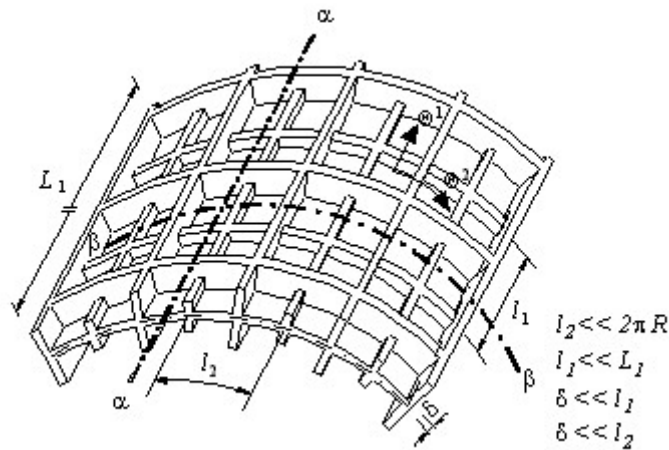
Figure 3. A fragment of the stiffened shell cross-section $\alpha - \alpha$
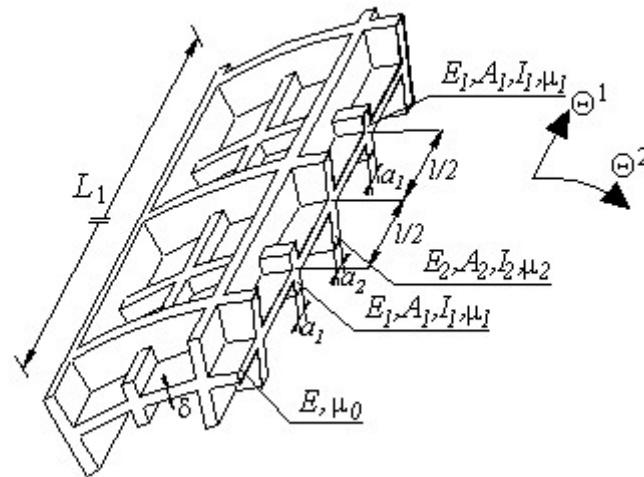
Figure 4. A fragment of the stiffened shell cross-section $\beta - \beta$

$$l << 2\pi R$$
$$a_1, a_2 << l$$
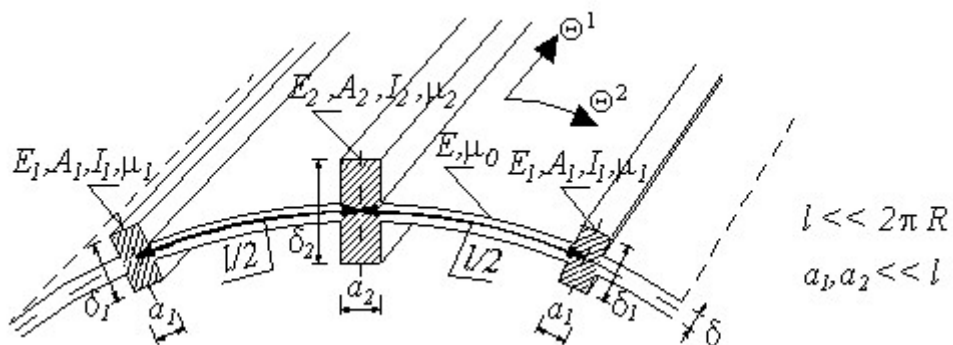
Denoting by a_1, a_2 the widths of the ribs (cf. Figures 3 and 4) we assume that $a_1, a_2 << l$ and hence the torsional rigidity of stiffeners can be neglected.

The tensile and bending rigidities of the stiffeners are constant. The rigidities of the shell are also constant and described by the components of the shell stiffness tensors $D_0^{\alpha\beta\gamma\delta}$ and $B_0^{\alpha\beta\gamma\delta}$ given by the known formulae, cf. [9]

$$D_0^{\alpha\beta\gamma\delta} = DH^{\alpha\beta\gamma\delta}, \qquad B_0^{\alpha\beta\gamma\delta} = BH^{\alpha\beta\gamma\delta}, \qquad (19)$$

where

$$D = E\delta/(1-v^2), \qquad B = E\delta^3/(12(1-v^2)),$$

(20)

$$H^{\alpha\beta\gamma\delta} = 0.5[a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma} + v(\in^{\alpha\gamma}\in^{\beta\delta} + \in^{\alpha\delta}\in^{\beta\gamma})]$$

with $a^{\alpha\gamma}$, $\in^{\alpha\gamma}$ as contravariant first midsurface tensor and Ricci bivector, respectively. After some manipulations we obtain the following expressions for the nonzero components of tensor $H^{\alpha\beta\gamma\delta}$:
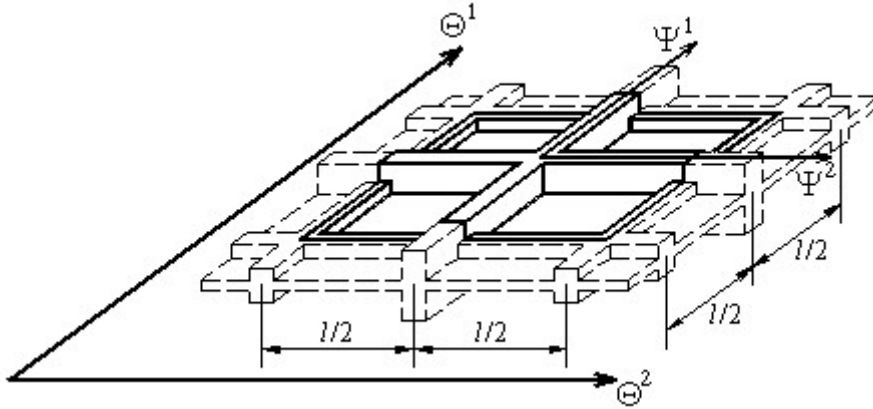
$$H^{1111} = H^{2222} = 1, \quad H^{1122} = H^{2211} = v, \quad H^{1212} = H^{1221} = H^{2121} = H^{2112} = (1-v)/2. \quad (21)$$

We define the square periodicity cell $\ddot{A}$ on $O\Theta^1\Theta^2$-plane by means of $\ddot{A} \equiv (-l/2, l/2) \times (-l/2, l/2)$, $\ddot{A}(\Theta) \equiv \ddot{A} + \Theta$, $\Theta \equiv (\Theta^1, \Theta^2) \in \Omega_{\ddot{A}}$, $\Omega_{\ddot{A}} := \{\Theta \in \Omega, \ddot{A}(\Theta) \in \Omega\}$. Setting $\Psi \equiv (\Psi^1, \Psi^2) \in \ddot{A}(\Theta)$, we assume that the cell Δ has two symmetry axes for $\Psi^1 = 0$ and $\Psi^2 = 0$. The cell $\ddot{A}$ is shown in Figure 5.

Figure 5. A periodicity cell on $O\Theta^1\Theta^2$-plane, $a_1, a_2 << l$

The periodically densely ribbed shell under consideration will be treated as a non-stiffened shell with constant thickness δ, made of a certain non-homogeneous material. Let us denote by $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$ and μ the stiffness tensors and mass density of this non-ribbed shell, respectively. The non-stiffened shell's tensile stiffnesses D^{1111}, D^{2222} and its bending stiffnesses B^{1111}, B^{2222} are l-periodic functions in Θ. Inside the cell $\ddot{A}$, these rigidities take the following form

$$D^{1111}(\Psi) = \begin{cases} D_0^{1111} = D & for \quad \Psi \in (-l/2, 0) \times (-l/2, 0) \cup (-l/2, 0) \times (0, l/2) \cup \\ & \cup (0, l/2) \times (-l/2, 0) \cup (0, l/2) \times (0, l/2), \\ E_1 A_1/2 & for \quad \Psi \in [-l/2, l/2] \times \{-l/2\} \cup [-l/2, l/2] \times \{l/2\}, \\ E_2 A_2 & for \quad \Psi \in [-l/2, l/2] \times \{0\}, \end{cases} \quad (22)$$

$$D^{2222}(\Psi) = \begin{cases} D_0^{2222} = D & for \quad \Psi \in (-l/2, 0) \times (-l/2, 0) \cup (-l/2, 0) \times (0, l/2) \cup \\ & \cup (0, l/2) \times (-l/2, 0) \cup (0, l/2) \times (0, l/2), \\ E_1 A_1/2 & for \quad \Psi \in \{-l/2\} \times [-l/2, l/2] \cup \{l/2\} \times [-l/2, l/2], \\ E_2 A_2 & for \quad \Psi \in \{0\} \times [-l/2, l/2], \end{cases} \quad (23)$$

$$B^{1111}(\Psi) = \begin{cases} B_0^{1111} = B & for \quad \Psi \in (-l/2, 0)\times(-l/2, 0)\cup(-l/2, 0)\times(0, l/2)\cup \\ & \quad \cup (0, l/2)\times(-l/2, 0)\cup(0, l/2)\times(0, l/2)\,, \\ E_1 I_1/2 & for \quad \Psi \in [-l/2, l/2]\times\{-l/2\}\cup[-l/2, l/2]\times\{l/2\}\,, \\ E_2 I_2 & for \quad \Psi \in [-l/2, l/2]\times\{0\}\,, \end{cases} \tag{24}$$

$$B^{2222}(\Psi) = \begin{cases} B_0^{2222} = B & for \quad \Psi \in (-l/2, 0)\times(-l/2, 0)\cup(-l/2, 0)\times(0, l/2)\cup \\ & \quad \cup (0, l/2)\times(-l/2, 0)\cup(0, l/2)\times(0, l/2)\,, \\ E_1 I_1/2 & for \quad \Psi \in \{-l/2\}\times[-l/2, l/2]\cup\{l/2\}\times[-l/2, l/2]\,, \\ E_2 I_2 & for \quad \Psi \in \{0\}\times[-l/2, l/2]\,, \end{cases} \tag{25}$$

Under assumption that the torsional rigidity of stiffeners is neglected, the remaining rigidities of the non-stiffened shell are constant and given by $D^{\alpha\beta\gamma\delta} = D_0^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta} = B_0^{\alpha\beta\gamma\delta}$ Taking into account expressions (19)-(21), these constant non-zero stiffnesses are given by

$$D^{1122} = D^{2211} = D\nu, \qquad D^{1212} = D^{1221} = D^{2121} = D^{2112} = D(1-\nu)/2\,,$$
$$B^{1122} = B^{2211} = B\nu, \qquad B^{1212} = B^{1221} = B^{2121} = B^{2112} = B(1-\nu)/2\,,$$

The mass density μ of the non-stiffened shell is l-periodic function in Θ and inside the cell $\ddot{A}$ is given by

$$\mu(\Psi) = \begin{cases} \mu_0 & for \quad \Psi \in (-l/2, 0)\times(-l/2, 0)\cup(-l/2, 0)\times(0, l/2)\cup \\ & \quad \cup (0, l/2)\times(-l/2, 0)\cup(0, l/2)\times(0, l/2)\,, \\ \mu_1 & for \quad \Psi \in [-l/2, l/2]\times\{-l/2\}\cup[-l/2, l/2]\times\{l/2\}\cup \\ & \quad \cup \{-l/2\}\times[-l/2, l/2]\cup\{l/2\}\times[-l/2, l/2]\,, \\ \mu_2 & for \quad \Psi \in [-l/2, l/2]\times\{0\}\cup\{0\}\times[-l/2, l/2] \end{cases} \tag{26}$$

Taking into account definition (1) we obtain for functions $D^{1111}(\Psi)$, $D^{2222}(\Psi)$, $B^{1111}(\Psi)$, $B^{2222}(\Psi)$, $\mu(\Psi)$ given above the following averaged values

$$\widetilde{D}^{1111} \equiv < D^{1111} >= D + (E_1 A_1 + E_2 A_2)/l\,, \quad \widetilde{D}^{2222} \equiv < D^{2222} >= D + (E_1 A_1 + E_2 A_2)/l\,,$$
$$\widetilde{B}^{1111} \equiv < B^{1111} >= B + (E_1 I_1 + E_2 I_2)/l\,, \quad \widetilde{B}^{2222} \equiv < B^{2222} >= B + (E_1 I_1 + E_2 I_2)/l\,, \tag{27}$$
$$\widetilde{\mu} \equiv < \mu >= \mu_0 + 2(\mu_1 + \mu_2)/l\,.$$

In order to analyze the problem of dynamic shell stability, we assume that the external forces f^β, f are equal to zero and that the shell is uniformly compressed in axial direction by the time-dependent forces $\overline{N}(t) \equiv \overline{N}^{11}(t)$; hence $\overline{N}^{12} = \overline{N}^{21} = \overline{N}^{22} = 0$. Moreover, the forces of inertia in directions tangential to the shell midsurface will be neglected.

Let the investigated problem be axisymmetric. For axisymmetric deformation $U_2 = Q_2^A = 0$ and the remaining unknowns U_1, Q_1^A, W, V^A are only the functions of the Θ^1-midsurface parameter.

For the sake of simplicity, we shall confine ourselves to the simplest form of the tolerance model in which $A=N=1$. Hence, we introduce only two l-periodic mode-shape functions $h(\mathbf{\Psi}) \equiv h^1(\mathbf{\Psi})$ and $g(\mathbf{\Psi}) \equiv g^1(\mathbf{\Psi})$, $\mathbf{\Psi} \in \ddot{A}(\mathbf{\Theta})$, which have to satisfy condition $<\mu\,h>=<\mu\,g>=0$ and the values of which are of order $5(l)$ and $5(l^2)$, respectively. Functions $h(\mathbf{\Psi})$, $g(\mathbf{\Psi})$ can be obtained as solutions to periodic eigenvalue problem for free vibrations on the cell given by equation (9) and hence they are referred to the lowest natural vibration modes related to the smallest free vibration frequencies in directions tangent and normal to the shell midsurface, respectively. Moreover, taking into account the symmetric form of the cell, cf. Figure 5, we assume that the mode-shape function $h(\mathbf{\Psi})$ is antisymmetric on the cell Λ while the mode-shape function $g(\mathbf{\Psi})$ is symmetric.

Taking into account the fact that, except for $D^{1111}, D^{2222}, B^{1111}, B^{2222}$, the components of the shell stiffness tensors $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$ are constant and baring in mind the symmetric form of the cell and the symmetric form of function $g(\mathbf{\Psi})$ as well as antisymmetric form of function $h(\mathbf{\Psi})$, it can be shown that only the following averages in (10) are different from zero: $\widetilde{D}^{\alpha\beta\gamma\delta}$, $\widetilde{B}^{\alpha\beta\gamma\delta}$, L^{A22}, K^{A11}, K^{A22}, C^{AB11}, C^{AB22}, S^{AB}, $\overline{L}^{AB}$, $\widetilde{\mu}$, $\overline{\mu}^{AB}$, $A,B=1$. It is evident that the aforementioned averages $\widetilde{D}^{\alpha\beta\gamma\delta}$ and $\widetilde{B}^{\alpha\beta\gamma\delta}$ are different from zero for the non-zero components of tensors $D^{\alpha\beta\gamma\delta}$ and $B^{\alpha\beta\gamma\delta}$. Moreover, under assumption that the external forces $f^{\beta} = f = 0$ and that the forces of inertia in directions tangent to M are neglected we have in (10): $\widetilde{\mu}^{AB} = \widetilde{P}^{A\beta} = \widetilde{P}^{A} = 0$, $A,B=1$.

Under assumption $A=B=N=1$, for the mentioned above non-zero averages including the non-tensorial indices A,B we introduce the following denotations

$$L^{22} \equiv L^{A22}, \quad K^{11} \equiv K^{A11}, \quad K^{22} \equiv K^{A22}, \quad C^{11} \equiv C^{AB11}, \quad C^{22} \equiv C^{AB22},$$
$$S \equiv S^{AB}, \quad \overline{L} \equiv \overline{L}^{AB}, \quad \overline{\mu} \equiv \overline{\mu}^{AB}, \qquad A,B=1. \tag{28}$$

We also denote $Q_1(\Theta^1,t) \equiv Q_1^1(\Theta^1,t)$, $V(\Theta^1,t) \equiv V^1(\Theta^1,t)$.

Bearing in mind the conditions and denotations given above we will derive below the formulae for boundaries of dynamic instability regions (i.e. the boundaries of regions of parametric resonance) for the considered biperiodic shell by using both the tolerance model given by Eqs.(11)-(13) and the homogenized model presented by Eqs.(17),(18.).

The tolerance model

Now, the governing equations (12),(13) of the tolerance model is separated into the independent equation for $Q_1(\Theta^1,t)$: $C^{11}Q_1 = 0$, which yields $Q_1 = 0$, and the system of three equations for macrodisplacements $U_1(\Theta^1,t)$, $W(\Theta^1,t)$ and fluctuation variable $V(\Theta^1,t)$

$$\widetilde{D}^{1111}U_{1,11} + Dv\,R^{-1}W_{,1} = 0,$$
$$DvR^{-1}U_{1,1} + \widetilde{B}^{1111}W_{,1111} + \widetilde{D}^{2222}R^{-2}W + \overline{N}W_{,11} + \widetilde{\mu}\,\ddot{W} + K^{11}V_{,11} - R^{-1}l^2L^{22}V = 0, \tag{29}$$
$$-R^{-1}l^2L^{22}W + K^{11}W_{,11} + \left(S + l^4\overline{L}\right)V + l^4\overline{\mu}\,\ddot{V} = 0,$$

where some terms depend explicitly on the period length $l = l_1 = l_2$; the averages $\widetilde{D}^{1111}$, $\widetilde{D}^{2222}$, $\widetilde{B}^{1111}$, $\widetilde{\mu}$ are defined by (27) and the remaining ones are given by (28) and (10).

It is easy to see, that all coefficients of the above equations are constant.

Separating variables Θ^1 and t, the solutions to Eqs.(29) satisfying boundary conditions for the simply supported shell on the edges $\Theta^1 = 0$, $\Theta^1 = L_1$ can be assumed in the form (see [9])

$$U_1(\Theta^1, t) = \sum_{m=1}^{\infty} T_m^U(t) \cos(\alpha_m \Theta^1), \qquad W(\Theta^1, t) = \sum_{m=1}^{\infty} T_m(t) \sin(\alpha_m \Theta^1),$$

$$V(\Theta^1, t) = \sum_{m=1}^{\infty} T_m^V(t) \sin(\alpha_m \Theta^1), \tag{30}$$

where $\alpha_m = m\pi / L_1$.

Substituting the right-hand sides of (30) into (29) and after some manipulations, the equation for function $T_m(t)$ is obtained

$$l^4 \overline{\mu}\, \widetilde{\mu}\, \frac{d^4}{dt^4} T_m(t) + l^4 \overline{\mu}\, \frac{d^2}{dt^2}\left[\left(\widetilde{B}^{1111}\alpha_m^4 + \frac{\widetilde{D}^{2222}}{R^2} - \frac{(Dv)^2}{R^2 \widetilde{D}^{1111}} - \overline{N}^{11}(t)\alpha_m^2\right)T_m(t)\right] +$$

$$+ (S + l^4 \overline{L})\widetilde{\mu}\, \frac{d^2}{dt^2} T_m(t) + \left[\left(S + l^4 \overline{L}\right)\left(\widetilde{B}^{1111}\alpha_m^4 + \frac{\widetilde{D}^{2222}}{R^2} - \frac{(Dv)^2}{R^2 \widetilde{D}^{1111}} - \overline{N}^{11}(t)\alpha_m^2\right) + \tag{31}$$

$$- \left(K^{11}\alpha_m^2\left(1 + \frac{l^2 L^{22}}{R\, K^{11}\alpha_m^2}\right)\right)^2\right]T_m(t) = 0$$

Because the shell under consideration satisfies the condition $l / L_1 \ll 1$, i.e. $\alpha_m l \ll 1$ and also $\delta / l \ll 1$ and $l / R \ll 1$, in the sequel the simplified form of Eq.(31) will be applied, in which terms $l^4 \overline{\mu}\, \frac{d^2}{dt^2}\left[\left(\widetilde{B}^{1111}\alpha_m^4 + \frac{\widetilde{D}^{2222}}{R^2} - \frac{(Dv)^2}{R^2 \widetilde{D}^{1111}} - \overline{N}^{11}(t)\alpha_m^2\right)T_m(t)\right]$ can be neglected as small compared to $(S + l^4 \overline{L})\widetilde{\mu}\, \frac{d^2}{dt^2} T_m(t)$ and the term $\frac{l^2 L^{22}}{R\, K^{11}\alpha_m^2}$ can be neglected as small compared to 1. Thus, the frequency equation (31) takes the following approximate form

$$l^4 \overline{\mu}\, \widetilde{\mu}\, \frac{d^4}{dt^4} T_m(t) + (S + l^4 \overline{L})\widetilde{\mu}\, \frac{d^2}{dt^2} T_m(t) +$$

$$+ \left[\left(S + l^4 \overline{L}\right)\left(\widetilde{B}^{1111}\alpha_m^4 + \frac{\widetilde{D}^{2222}}{R^2} - \frac{(Dv)^2}{R^2 \widetilde{D}^{1111}} - \overline{N}^{11}(t)\alpha_m^2\right) - \left(K^{11}\right)^2 \alpha_m^4\right]T_m(t) = 0 \tag{32}$$

We assume that the compressive axial forces $\overline{N} \equiv \overline{N}^{11}(t)$ are given by

$$\overline{N}(t) = \overline{N}_a + \overline{N}_b \cos(pt) \tag{33}$$

where p is the oscillation frequency of these forces and $\overline{N}_a, \overline{N}_b$ are constant.

Let us denote

$$\eta_m \equiv \widetilde{B}^{1111} + \widetilde{D}^{2222} R^{-2}\alpha_m^{-4} - (Dv)^2 (\widetilde{D}^{1111})^{-1} R^{-2}\alpha_m^{-4}$$

and then introduce the following formulae

$$\omega_m^2 \equiv \alpha_m^4 (\tilde{\mu})^{-1} \Big[\eta_m - (K^{11})^2 (S + l^4 \overline{L})^{-1} \Big], \qquad \omega_*^2 \equiv (S + l^4 \overline{L})(l^2 \overline{\mu})^{-1},$$

$$\overline{N}_{cr,m} \equiv \alpha_m^2 \Big[\eta_m - (K^{11})^2 (S + l^4 \overline{L})^{-1} \Big], \qquad \Omega_m^2 \equiv \omega_m^2 \Big[1 - \overline{N}_a (\overline{N}_{cr,m})^{-1} \Big], \qquad (34)$$

$$2 \breve{\mu}_m \equiv \overline{N}_b (\overline{N}_{cr,m} - \overline{N}_a)^{-1},$$

where ω_m and ω_* are the *m-th lower* and *"additional" higher free vibration frequencies*, respectively, $\overline{N}_{cr,m}$ is the *m*-th static critical force, Ω_m is the *m*-th free vibration frequency of the shell subjected to an axial force $\overline{N}_a < \overline{N}_{cr,m}$ and $\breve{\mu}_m$ is the modulation factor.

Using these formulae, the frequency equation (32) can be transformed into

$$\frac{d^4 T_m}{dt^4} + \omega_*^2 \frac{d^2 T_m}{dt^2} + \omega_*^2 \Omega_m^2 \big[1 - 2\breve{\mu}_m \cos(pt) \big] T_m = 0 \qquad (35)$$

The above equation is a starting point of the analysis of dynamic stability of the considered shell in the framework of the non-asymptotic model. All parameters in this equation depend on the period length *l* and hence it makes it possible to investigate the length-scale effect on the parametric vibrations and dynamical stability of periodic shells. It must be emphasized that the obtained fourth order ordinary differential equation (35) is a certain generalization of the known Mathieu equation; it takes the form of the Mathieu equation provided that in (35) the length-scale effect is neglected.

The analysis of dynamic stability leads to the determination of the instability regions (parametric resonance regions) on the $(p/\Omega_m, \mu_m)$-plane, cf. [9]. Thus, applying a procedure similar to that used for the investigation of the Mathieu equation, we will determine the instability regions for solutions to equation (35). Within the resonance regions vibrations grow up in an unlimited way as $t \to \infty$. Outside and at the boundaries of the resonance regions there exist periodic solutions to equation (35) with the parametric excitation periods $\breve{T}_p = 2\pi / p$ and $2\breve{T}_p$.

Following [9], the solution to equation (35) with a period $2\breve{T}_p$, related to vibrations of the *m*-th harmonics of the series (30)$_2$, can be assumed in the form

$$T_m(t) = \sum_{k=1,3,5}^{\infty} \left(a_{mk} \sin \frac{kpt}{2} + b_{mk} \cos \frac{kpt}{2} \right), \qquad m = 1,2,... \qquad (36)$$

Substituting (36) into (35) and after comparing the coefficients of pertinent trigonometric functions to zero, we obtain two homogeneous, infinite systems of linear algebraic equations

$$\left[1 + \breve{\mu}_m - \left(\frac{p}{2\Omega_m} \right)^2 (1 - \xi^2) \right] a_{m1} - \breve{\mu}_m \, a_{m3} = 0 \,,$$

$$(37)$$

$$\left[1 - \left(\frac{kp}{2\Omega_m} \right)^2 (1 - k^2 \xi^2) \right] a_{mk} - \breve{\mu}_m \big(a_{m\,k-2} + a_{m\,k+2} \big) = 0 \quad, \quad k = 3,5,7,....$$

$$\left[1 + \breve{\mu}_m - \left(\frac{p}{2\Omega_m} \right)^2 (1 - \xi^2) \right] b_{m1} - \breve{\mu}_m \, b_{m3} = 0 \,,$$

$$(38)$$

$$\left[1 - \left(\frac{kp}{2\Omega_m} \right)^2 (1 - k^2 \xi^2) \right] b_{mk} - \breve{\mu}_m \big(b_{m\,k-2} + b_{m\,k+2} \big) = 0 \quad, \quad k = 3,5,7,....$$

where $\xi \equiv \dfrac{p}{2\omega_*}$; $\xi = 0$ for $l \to 0$.

For sufficiently small values of modulation factor ($\breve{\mu}_m \ll 1$) and for the basic, largest parametric resonance region which occurs in the vicinity of the frequency $p = 2\Omega_m$, the characteristic determinants of systems (37) and (38) can be approximated by the first components of relations (37)$_1$ and (38)$_1$. In this case, for every $m=1,2,\ldots$, on the $(p/\Omega_m, \mu_m)$-plane we obtain the boundaries of the first instability region (resonance region) given by

$$\left(\frac{p}{2\Omega_m}\right)^2 \approx \frac{1+\breve{\mu}_m}{1-\xi^2}, \qquad \left(\frac{p}{2\Omega_m}\right)^2 \approx \frac{1-\breve{\mu}_m}{1-\xi^2}. \tag{39}$$

Solutions (39) take into account the length-scale effect; the free vibration frequency Ω_m, the modulation factor $\breve{\mu}_m$ and the additional higher free vibration frequency ω_* depend on the period length $l \equiv l_1 = l_2$. We have also obtained the extra condition for the oscillation frequency p of axial compressive forces: $p \neq 2\omega_*$. For $l \to 0$, expressions (39) take the form known in literature, cf.[9].

Now, we are looking for solution to equation (35) with a period $\breve{T}_p$. Following [9], this solution related to vibrations of m-th harmonics of the series (30)$_2$ can be assumed in the form

$$T_m(t) = b_{m0} + \sum_{k=2,4,6}^{\infty} \left(a_{mk} \sin \frac{k\,pt}{2} + b_{mk} \cos \frac{k\,pt}{2} \right), \qquad m = 1,2,\ldots \tag{40}$$

Substituting (40) into (35) and after comparing the coefficients of pertinent trigonometric functions to zero, we obtain two homogeneous, infinite systems of linear algebraic equations

$$b_{m0} - \breve{\mu}_m b_{m2} = 0,$$

$$\left[1 - \left(\frac{p}{\Omega_m}\right)^2 \left(1 - \tilde{\xi}^2\right)\right] b_{m2} - \breve{\mu}_m (2b_{m0} + b_{m4}) = 0, \tag{41}$$

$$\left[1 - \left(\frac{k\,p}{2\Omega_m}\right)^2 \left(1 - k^2\,\tilde{\xi}^2\right)\right] b_{m\,k} - \breve{\mu}_m \left(b_{m\,k-2} + b_{m\,k+2}\right) = 0, \quad k = 4,6,8,\ldots$$

$$\left[1 - \left(\frac{p}{\Omega_m}\right)^2 \left(1 - \tilde{\xi}^2\right)\right] a_{m2} - \breve{\mu}_m a_{m4} = 0,$$

$$\tag{42}$$

$$\left[1 - \left(\frac{k\,p}{2\Omega_m}\right)^2 \left(1 - k^2\,\tilde{\xi}^2\right)\right] a_{m\,k} - \breve{\mu}_m \left(a_{m\,k-2} + a_{m\,k+2}\right) = 0, \quad k = 4,6,8,\ldots$$

where $\tilde{\xi} \equiv \dfrac{p}{\omega_*}$; $\tilde{\xi} = 0$ for $l \to 0$.

For sufficiently small values of modulation factor ($\breve{\mu}_m \ll 1$) and for the second parametric resonance region which occurs in the vicinity of the frequency $p = \Omega_m$, the characteristic determinants related to systems (41) and (42) can be restricted to two rows and two columns. Hence, for each value $m=1,2,\ldots$, on the $(p/\Omega_m, \mu_m)$-plane we obtain the boundaries of the second instability region given by

$$\left(\frac{p}{\Omega_m}\right)^2 \approx \frac{1 - 2(\breve{\mu}_m)^2}{1 - \tilde{\xi}^2}, \qquad \left(\frac{p}{\Omega_m}\right)^2 \approx \frac{1 + \dfrac{1}{3}(\breve{\mu}_m)^2}{1 - \tilde{\xi}^2}. \tag{43}$$

Results (43) depend on the cell size and for $l \rightarrow 0$ take the classical form known in literature. Moreover, we have obtained the extra condition for the oscillation frequency p of axial compressive forces: $p \neq \omega_*$.

The homogenized model

In order to evaluate obtained results, let us consider the above problem within the homogenized (i.e. asymptotic) model. From Eqs.(29), after neglecting the terms of orders $5(l^2)$ and $5(l^4)$, we obtained the following governing relations of the homogenized model

$$\widetilde{D}^{1111} U_{,11} + Dv\, R^{-1} W_{,1} = 0,$$

$$DvR^{-1} U_{,1} + \widetilde{B}^{1111} W_{,1111} + \widetilde{D}^{2222} R^{-2} W + \overline{N} W_{,11} + \widetilde{\mu}\, \ddot{W} + K^{11} V_{,11} = 0. \qquad (44)$$

$$K^{11} W_{,11} + SV = 0.$$

The obtained above model is not able to describe the length-scale effect on the overall shell stability being independent of the period length l.

The solutions to Eqs.(44) can be assumed in the form (30). Substituting (30) into (44) and setting

$$\omega_{0\,m}^2 \equiv \alpha_m^4 (\widetilde{\mu})^{-1} [\eta_m - (K^{11})^2 S^{-1}], \qquad \overline{N}_{0\,cr,m} \equiv \alpha_m^2 [\eta_m - (K^{11})^2 S^{-1}],$$

$$\Omega_{0m}^2 \equiv \omega_{0m}^2 [1 - \overline{N}_a (\overline{N}_{0\,cr,m})^{-1}], \qquad 2\breve{\mu}_{0\,m} \equiv \overline{N}_b (\overline{N}_{0\,cr,m} - \overline{N}_a)^{-1}, \qquad (45)$$

we arrive at the following frequency equation

$$\frac{d^2 T_m}{dt^2} + \Omega_{0\,m}^2 \left[1 - 2\breve{\mu}_{0\,m} \cos(pt) \right] T_m = 0 \qquad (46)$$

It is easy to see that all parameters of the above equation are independent of the cell size and that in the framework of the asymptotic model it is not possible to determine the additional higher free vibration frequency, caused by the periodic structure of the shell. It is also easy to see that the equation (46) has a form of the known Mathieu equation, which describes dynamic stability and parametric vibrations of different structures, cf. [9].

Using the procedure for the determination of the instability region boundaries describes in the previous subsection, we obtain the classical formulae

– for the first instability region (vibrations with period $2\breve{T}_p$)

$$\left(\frac{p}{2\Omega_{0\,m}} \right)^2 \approx 1 + \breve{\mu}_{0\,m}, \qquad \left(\frac{p}{2\Omega_{0\,m}} \right)^2 \approx 1 - \breve{\mu}_{0\,m}. \qquad (47)$$

– for the second instability region (vibrations with period $\breve{T}_p$)

$$\left(\frac{p}{\Omega_{0\,m}} \right)^2 \approx 1 - 2(\breve{\mu}_{0\,m})^2, \qquad \left(\frac{p}{\Omega_{0\,m}} \right)^2 \approx 1 + \frac{1}{3}(\breve{\mu}_{0\,m})^2. \qquad (48)$$

In the next subsection a comparison of the results obtained in the preceding subsections will be presented.

Comparison of results

First of all, let us compare the lower free vibration frequency ω_m^2 given by $(34)_1$ and the static critical force $\overline{N}_{cr,m}$ defined by $(34)_3$, which have been obtained in the framework of the tolerance model, with the free vibration

frequency $\omega_{0\,m}^2$ given by (45)$_1$ and the static critical force $\overline{N}_{0\,cr,m}$ determined by (45)$_2$ obtained from the homogenized model. To this end, let us denote $\varepsilon \equiv l^4$, where the constant ε can be treated as a small parameter. It is easy to show that using this notation and then representing the right-hand sides of formulae (34)$_1$ and (34)$_3$ in the form of the power series with respect to ε, we obtain

$$\omega_m^2 \equiv \alpha_m^4 (\tilde{\mu})^{-1}[\eta_m - (K^{11})^2 S^{-1}] + 5(\varepsilon), \qquad \overline{N}_{cr,m} \equiv \alpha_m^2 [\eta_m - (K^{11})^2 S^{-1}] + 5(\varepsilon). \tag{49}$$

Taking into account expressions (45)$_1$ and (45)$_2$, we arrive finally at the following interrelations

$$\omega_m^2 = \omega_{0\,m}^2 + \, 5(l^4), \qquad\qquad \overline{N}_{cr,m} = \overline{N}_{0\,cr,m} + 5(l^4). \tag{50}$$

It means that the differences between the values of squares of the m-th free vibration frequencies ω_m^2 and $\omega_{0\,m}^2$ obtained within the framework of the tolerance and homogenized models, respectively, as well as between the values of the m-th critical forces $\overline{N}_{cr,m}$ and $\overline{N}_{0\,cr,m}$ obtained from both the models under consideration are negligibly small. Thus, *the effect of the period length l on the free vibrations and critical forces of the shell under consideration can be neglected*; i.e.

$$\omega_m^2 \cong \omega_{0\,m}^2 \, , \qquad\qquad \overline{N}_{cr,m} \cong \overline{N}_{0\,cr,m} \, . \tag{51}$$

Next, substituting the right-hand sides of (51) into expressions (34)$_4$, (34)$_5$ and comparing the obtained results with formulae (45)$_3$, (45)$_4$, respectively, we conclude that, in the framework of tolerance, the m-th free vibration frequency Ω_m of the shell subjected to axial forces $\overline{N}_a < \overline{N}_{cr,m}$ and the modulation factor $\breve{\mu}_m$ calculated from the tolerance model are equal to the m-th free vibration frequency $\Omega_{0\,m}$ and the modulation factor $\breve{\mu}_{0\,m}$, respectively, obtained in the framework of the asymptotic model, i.e

$$\Omega_m^2 \cong \Omega_{0\,m}^2 \, , \qquad\qquad \breve{\mu}_m \cong \breve{\mu}_{0\,m} \, . \tag{52}$$

In this case, substituting the right-hand sides of (52) into (39) and (43), for every values of $m=1,2,\ldots$, we obtain the following formulae for the boundaries

– of the first instability region (vibrations with period $2\breve{T}_p$)

$$\left(\frac{p}{2\Omega_{0\,m}}\right)^2 \approx \frac{1+\breve{\mu}_{0\,m}}{1-\xi^2} \, , \qquad \left(\frac{p}{2\Omega_{0\,m}}\right)^2 \approx \frac{1-\breve{\mu}_{0\,m}}{1-\xi^2} \, , \qquad \xi \equiv \frac{p}{2\omega_*} , \tag{53}$$

– of the second instability region (vibrations with period $\breve{T}_p$)

$$\left(\frac{p}{\Omega_{0\,m}}\right)^2 \approx \frac{1-2(\breve{\mu}_{0\,m})^2}{1-\widetilde{\xi}^2} \, , \qquad \left(\frac{p}{\Omega_{0\,m}}\right)^2 \approx \frac{1+\frac{1}{3}(\breve{\mu}_{0\,m})^2}{1-\widetilde{\xi}^2} \, , \qquad \widetilde{\xi} \equiv \frac{p}{\omega_*} , \tag{54}$$

within the framework of the tolerance model; the period length l is contained in the additional higher free vibration frequency ω_* given by (34)$_2$.

Comparing the obtained above expressions (53) and (54) derived from the tolerance non-asymptotic model with the corresponding expressions (47) and (48), respectively, obtained from the homogenized (i.e. asymptotic) one, we conclude that the differences between results obtained from both the models under consideration increase with

increasing the parameters ξ and $\tilde{\xi}$, i.e. *with increasing values of the oscillation frequency p of the axial compressive forces* $\overline{N}(t) \equiv \overline{N}^{11}(t)$ and *with decreasing the higher free vibration frequency* ω_*. The frequency ω_* decreases with increasing the values of the period length *l*, the growth of which is restricted by the condition $l << \min(L_1, R)$. It can also be observed that because the expressions (53) and (54) describe the boundaries of parametric resonance regions which occur in the vicinity of the oscillation frequency $p = 2\Omega_{0\,m}$ and $p = \Omega_{0\,m}$, respectively, then the frequency *p* grows with increasing the free vibration frequency $\Omega_{0\,m}$, i.e. with increasing *m*=1,2,…. For every investigated shell, the upper limits of values of parameters ξ and $\tilde{\xi}$ have to be determined experimentally. It has to be emphasized that, because the values of the additional free vibration frequency ω_* are very large then the situation in which the oscillation frequency *p* of the axial compressive forces is very close to the frequency ω_* (in this case the length-scale effect would be very big) is impossible from the physical point of view.

Let us remember that the results obtained here have a physical sense, provided that the upper critical forces are sufficient from the point of view of calculations made for the problem of determining the boundaries of instability regions in biperiodically densely stiffened cylindrical shells under consideration.

CONCLUSIONS

Summarizing the results obtained in this section the following conclusions can be formulated:

- Contrary to homogenized (asymptotic) model, the proposed non-asymptotic one describes the effect of the period lengths on the shell dynamic shell stability.
- In the framework of the non-asymptotic tolerance model proposed in this contribution, *the fundamental lower and additional higher free vibration frequencies* can be derived. Differences between these lower free vibration frequencies and free vibration frequencies obtained from the asymptotic model are negligibly small. On the other hand, the higher free vibration frequency, caused by a periodic structure of the stiffened shell, cannot be determined using the homogenized (i.e. asymptotic) model.
- Taking into account the effect of the period lengths on dynamic stability of thin periodic shells we arrive at *the fourth-order ordinary differential equation for the unknown function of time coordinate, which can be treated as a certain generalization of the known Mathieu equation*. It reduces to the Mathieu equation provided that the period lengths are neglected. On the contrary, within the homogenized model the known Mathieu equation is obtained.
- The differences between boundaries of the first and second dynamic instability regions, obtained from the tolerance model and the asymptotic one, increase with increasing values of the oscillation frequency *p* of the axial compressive forces and with increasing values of the period lengths. For high values of *p* the length-scale effect plays a crucial role and cannot be neglected.

FINAL REMARKS

The subject-matter of this contribution is a thin linear-elastic cylindrical shell having a periodic structure (a periodically varying thickness and/or periodically varying elastic and inertial properties) in both directions tangent to the undeformed shell midsurface *M*. Shells of this kind are termed *biperiodic*. Moreover, it is assumed that the biperiodic cylindrical shells, being objects of our considerations, are composed of a very large number of identical elements and every such element is treated as a shallow shell. It means that the periods of inhomogeneity are very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface in the periodicity direction. This biperiodic structure of cylindrical shells considered here can be related to the periodically spaced dense system of ribs as shown in Figure 1.

For the biperiodic cylindrical shells the known governing equations of the Kirchhoff-Love shell theory involve periodic highly oscillating and non-continuous coefficients. Hence, in most cases direct application of these equations to analyze engineering problems in periodic shells is very complicated, particularly from the computational viewpoint. That is why the aim of this contribution was to propose *a new non-asymptotic model of biperiodic cylindrical shells for problems of dynamics and dynamical stability,* which has constant coefficients and hence can be applied as a proper analytical tool for investigations of engineering problems in the shell under considerations. Moreover, the proposed model takes into account the effect of periodicity cell size on the global

shell dynamics and dynamical stability as well as stationary stability, called *the length-scale effect*, which is neglected in the known homogenized models derived by asymptotic methods.

In order to derive the model equations the *tolerance averaging procedure* given in [22], has been applied to governing equations of the Kirchhoff-Love second-order shell theory for thin linear-elastic cylindrical shells, i.e. to equations (2)-(4). The proposed averaged non-asymptotic model called *the tolerance model of dynamic and dynamical stability problems for biperiodic cylindrical shells* is governed by the constitutive relations (11), by the system of differential equations (12),(13) with constant coefficients and by the approximation formula (14) for the total shell displacements. The basic unknowns are: the *macrodisplacements* U_α , W and the *fluctuation variables* Q_α^A, V^A, $A = 1,2,...,N$, which have to be *slowly-varying functions* with respect to the cell and certain tolerance system. This requirement imposes certain restrictions on the class of problems described by the model under consideration. It can be observed that the constant coefficients in (12),(13) depend on the period lengths and hence describe the effect of a cell size on the overall behavior of the biperiodic shell.

The boundary conditions for macrodisplacements are the same as in the classical shell theory. The fluctuation variables are governed by the system of ordinary differential equations involving only time derivatives and hence there are no extra boundary conditions for these functions, and that is why they play the role of kinematic internal variables.

In order to obtain the governing equations *the mode-shape (shape) functions* h^A, g^A, $A = 1,2,...,N$, should be derived from the periodic finite element method discretization of the cell or obtained as solutions to periodic eigenvalue problem on the cell given by equations (9). This eigenvalue problem describes free periodic vibrations of the cell, and hence the eigenfunctions h^A, g^A, A=1,2,..,N, represent the expected forms of the oscillating part of free vibration modes of the periodicity cell. Moreover, in most problems the analysis is restricted to the simplest case *N*=1 in which we take into account only the lowest natural vibration modes (in directions tangent and normal to the shell midsurface) related to the smallest free vibration frequencies.

It has to be emphasized that the tolerance model (11)-(13) has been derived in the framework of the geometrically linear stability theory for thin linear-elastic Kirchhoff-Love type shells. That is why, this model can be applied to analyze the problems of dynamic shell stability, provided that the upper critical forces are sufficient from the point of view of calculations made for solving those problems.

Let us note that the model presented here can be treated as a certain generalization of the models given in [19, 20], because makes it possible to analyze not only free and forced vibrations and static critical forces but also parametric vibrations and dynamical stability of biperiodic cylindrical shells. However, it has to be emphasized that the biperiodic shells are special cases of those with a periodic structure along one direction tangent to the shell midsurface (called uniperiodic) and hence the model proposed here cannot be applied to investigate the problems of uniperiodic shells.

In this paper, it has also been shown that the model without the length-scale effect, called homogenized or asymptotic, is a special case of the tolerance one.

The derived both the tolerance and asymptotic models have been used in this contribution to investigate the effect of the period lengths on the dynamic stability of closed, simply supported, biperiodically densely stiffened cylindrical shells under time-dependent axial compressive forces. In the framework of the tolerance model we have obtained *the fourth-order ordinary differential equation for the unknown function of time coordinate, which can be treated as a certain generalization of the known Mathieu equation*. It reduces to the Mathieu equation provided that the period lengths are neglected. On the contrary, within the homogenized model the known Mathieu equation is obtained. It has been shown that the differences between the boundaries of the dynamic instability regions obtained from the generalized Mathieu equation and from the classical Mathieu equation are large, particularly for high values of oscillation frequency of the axial compressive forces. It means that *the length-scale effect on the dynamic shell stability cannot be neglected*.

Moreover, in the framework of the non-asymptotic model proposed here, not only *the fundamental lower but also the additional higher free vibration frequencies* can be derived and analyzed. These *higher free vibration frequencies* depend on the period lengths and cannot be derived from the asymptotic models. On the other hand, the effect of the period lengths on the *lower free vibration frequencies* is negligibly small and hence they can be approximated by similar frequencies derived from the asymptotic models.

Problems related to various applications of the proposed equations (11)-(13) to dynamics and dynamical stability of biperiodic cylindrical shells as well as the possible generalizations of these equations are reserved for separate papers. Determination of the mode-shape functions from periodic eigenvalue problem given by (9) is also reserved for a separate paper.

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