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# MODELLING OF FUNCTIONALLY GRADED LAMINATES REVISITED

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## ABSTRACT

Optimization of structural elements met both in civil and mechanical engineering leads to the concept of a material with characteristics smoothly varying in a certain direction (functionally graded materials). An example of this material is a laminate having macroscopic properties varying in space (functionally graded laminate). The aim of this note is to propose a revisited version of the mathematical model for analyzing the effect of interlaminar microdefects on the macroscopic response of a structural element. It is assumed that the distribution of these defects is represented by its mean density per unit area of every interface.

**Key words:** functionally graded laminates, interlaminar microdefects, elastodynamics.

## INTRODUCTION

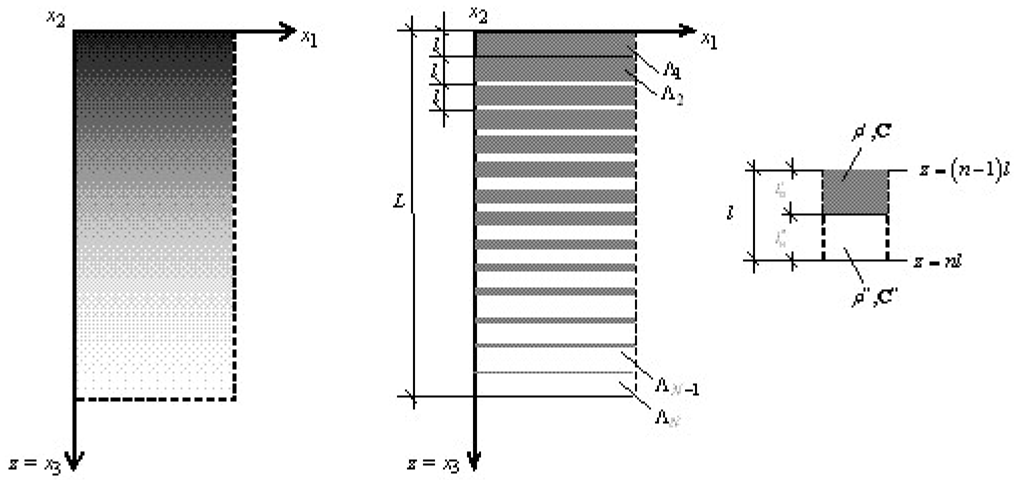
The object of analysis are two-phase linear-elastic multilayered laminated media with a gradual variation of macroscopic properties in the direction normal to the layering (functionally graded laminates, FGL). We assume that every layer has the same thickness and consists of two homogeneous laminae having different thicknesses in different layers. A fragment of the FGL on the macro and micro level is shown in Fig. 1. The FGL under consideration represents a special case of a functionally graded material (FGM) well studied in the recent literature (cf. [2] and the extensive list of references therein).

The purpose of this contribution is to formulate a mathematical model for the analysis of the effect of interlaminar microdefects both on macro- and micro- behavior of FGL. The above problem was discussed recently in [5]. However, the approach presented in the above note was not quite correct. That is why in this contribution we present a revisited modelling version of FGL with interlaminar defects. Considerations will be carried out in the framework of the linear elasticity theory. The lamina materials are assumed to have elastic symmetry planes parallel to the lamina interfaces. Distribution of microdefects on the interfaces of FGL is assumed to be continuous and represented by the known mean density.

## PRELIMINARIES

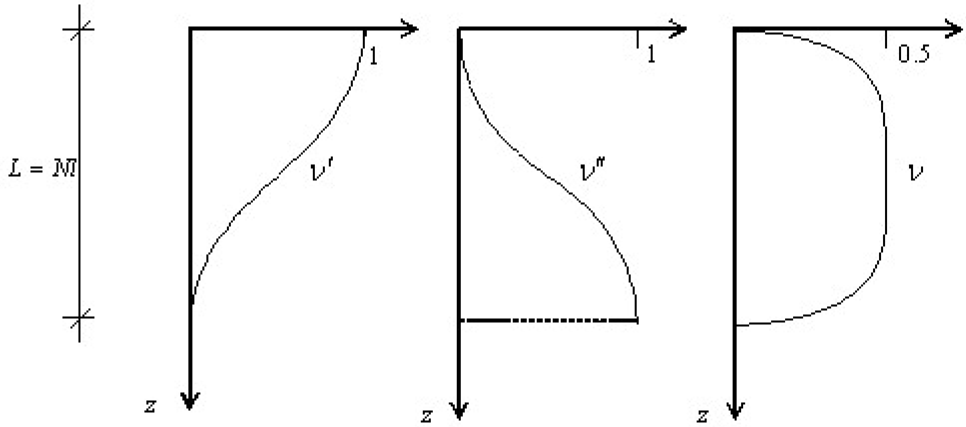
Let $0x_1x_2x_3$ stand for the orthogonal cartesian coordinate system in the physical space and $\Omega \times (0, L)$, $\Omega \subset R^2$, be the region in this space occupied by the FGL in the reference configuration. The $x_3$-axis is directed perpendicular to the lamina interfaces. By $\mathbf{x} = (x_1, x_2)$ we denote an arbitrary point on the plane region $\Omega$ and set $z = x_3$. The thickness of every layer is assumed to be constant and will be denoted by *l.* Let *N* be the number of layers and assume that $N^{-1} << 1$ (cf. Fig. 1). Hence $L = Nl$ is the thickness of FGL. Thicknesses of laminae constituting the *n*-th layer, *n*=1,…,*N*, are denoted by $l'_n$, $l''_n$. By $\rho'$, $\rho''$ and $\mathrm{X}'$, $\mathrm{X}''$ we denote mass densities and tensors of elastic moduli in every pair of adjacent laminae, respectively (cf. Fig. 1).

**Fig. 1. Fragments of FGL on the macro- and micro-structural level together with a scheme of the *n*-th layer, *n*=1,..,*N***




The main assumption is that there exist smooth functions $\nu'(\cdot)$, $\nu''(\cdot)$ defined on $[0, L]$ representing a distribution of the mean volume fractions of lamina materials. It means that $\nu'(z_n) = l'_n / l$, $\nu''(z_n) = l''_n / l$ for some $z_n \in [(n-1)l, nl]$, *n*=1,…,*N* and $\nu'(z) + \nu''(z) = 1$, $z \in [0, L]$.We also introduce function $\nu = \sqrt{\nu'\nu''}$ defined on $[0, L]$ which will be referred to as the mean phase distribution. Diagrams of mean volume fractions $\nu'(\cdot)$, $\nu''(\cdot)$ and phase distribution function $\nu = \sqrt{\nu'\nu''}$ are shown in Fig.2.

**Fig. 2. Diagrams of mean volume fractions $\nu'$, $\nu''$ and phase distribution function $\nu = \sqrt{\nu'\nu''}$**



*Notations.* Subscripts $\alpha$, $\beta$, $\gamma$, $\delta$ run over the sequence 1, 2 and subscripts *i, j, k, l* over 1, 2, 3, summation convention holds. By $\partial_\alpha f$ and $\partial_k f$ we denote partial derivatives of function $f(x_1, x_2, x_3)$ with respect to $x_\alpha$ and $x_k$, respectively. The time coordinate is denoted by *t* and the time differentiation by the overdot. We introduce gradient operators $\nabla = (\partial_1, \partial_2, \partial_3)$ and $\overline{\nabla} = (\partial_1, \partial_2, 0)$. Throughout the considerations the tensor notation is used with "dot" and "double dot" as the scalar and the double scalar products, respectively, and with $\otimes$ as the tensor product. Setting $\mathbf{i_1} = (1,0,0)$, $\mathbf{i}_2 = (0,1,0)$, $\mathbf{i}_3 = (0,0,1)$ we also introduce denotation $\overline{\mathrm{X}} = C_{\alpha\beta\gamma\delta}\mathbf{i}_\alpha \otimes \mathbf{i}_\beta \otimes \mathbf{i}_\gamma \otimes \mathbf{i}_\delta$, where $\mathrm{X}$ is a tensor of elastic moduli.

## CONSTITUTIVE EQUATIONS

The modelling approach to the problem under analysis is based on that used in [3] and will be recalled below. We shall assume that the distribution of interlaminar microdefects is known *a priori* and represented by its constant mean density $\chi$ per unit area of every interface between adjacent laminae, $\chi \in [0,1)$. Moreover, let $\eta$ stand for a mean density of open microdefects, $\eta \in [0, \chi]$ and $\varphi(\cdot)$ be the convex function defined on *R* given by

$$\varphi(e) = \begin{cases} 0 & if \quad e \leq 0 \\ \chi e & if \quad e > 0 \end{cases}$$

Function $\varphi(\cdot)$ was introduced in [3] and referred to as the potential of open microdefects. The length dimensions of every single microdefects are negligibly small as compared to the thickness of laminae. Hence the continuous distribution of microdefects is introduced. Under this assumption, following [3] we shall postulate the relation between strain tensor **E** and stress tensor **S** for an arbitrary lamina in the form

$$\mathbf{S} = \left[(1-\eta)\mathrm{X} + \eta\overline{\mathrm{X}}\right] : \mathbf{E} \tag{1}$$

The linear strain-stress relation (1) have to be considered together with nonlinear relation

$$\eta \in \partial\varphi(\mathbf{i}_3 \cdot \mathbf{E} \cdot \mathbf{i}_3) \tag{2}$$

where $\partial$ is the subdifferential of the convex function $\varphi(\cdot)$. For particulars the reader is referred to [3].

## FORMULATION OF THE MODELLING PROBLEM

Constitutive relations (1), (2), the equations of motion and the linearized strain-displacement relations constitute the system of field equations for elastodynamics of composites under consideration. These equations have to be considered together with the stress continuity conditions on every interface between laminae. Due to the large number of interfaces the aforementioned relations cannot be treated as a proper mathematical tool for the analysis of special problems. That is why the primary purpose of this contribution is to formulate an averaged (macroscopic) mathematical model of the FGL under consideration which is free of the above drawback. To this end we shall adapt some concepts and assumptions of the known tolerance averaging technique which so far has been used to the mathematical modelling of composites and laminates with a periodic structure (cf. [4] and the references therein). Basic concepts and assumptions of the tolerance averaging technique will be recall in the subsequent section.

## AUXILIARY CONCEPTS AND MODELLING ASSUMPTIONS

We begin with some auxiliary concepts. Function $F \in C^1([0, L])$ of argument $x_3$, which can also depend on $\mathbf{x} = (x_1, x_2)$ and *t*, will be called slowly varying (in every interval of the length *l*, $l << L$, and with a tolerance $\varepsilon$, $0 < \varepsilon << 1$), if functions $l\partial_3 F$ and $O(\varepsilon F)$ are of the same order for $\varepsilon \to 0$. It means that increments of an arbitrary slowly-varying function *F* in every interval $(-l/2 + x_3, l/2 + x_3)$ can be neglected when compared to the maximum value of $\left|F(x_3)\right|$, $x_3 \in [0,L]$. If this condition holds also for all partial derivatives of $F$ which occur in model equations then we shall write $F \in SV_{\varepsilon}(l)$, where $\varepsilon$ is called a tolerance parameter.

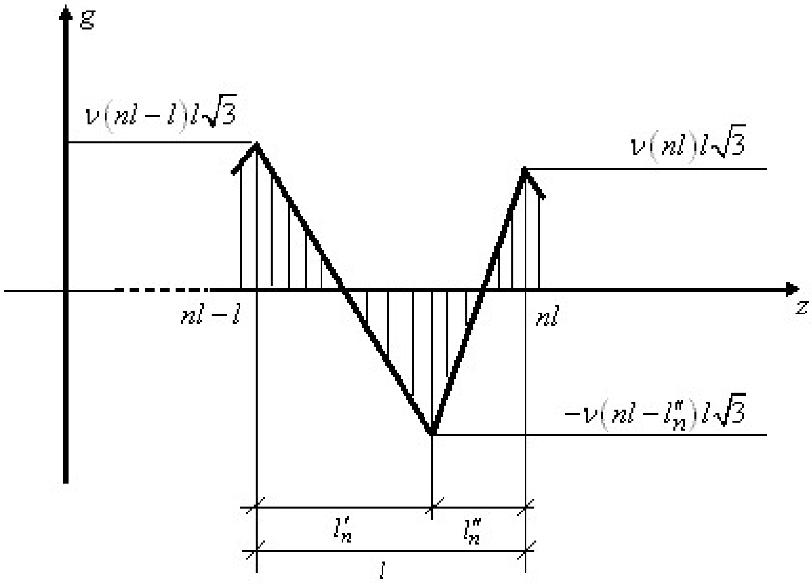
The mean value of an arbitrary integrable function $f$ defined in $(0,L)$ will be denoted by

$$\langle f\rangle(z)=\frac{1}{l}\int_{z-\frac{l}{2}}^{z+\frac{l}{2}} f(\zeta)d\zeta\,, \qquad z\in\left[\frac{l}{2},L-\frac{l}{2}\right]$$

Function $f$ can also depend on $\mathbf{x}=(x_1,x_2)$ and time $t$.

Let $g:[0,L]\rightarrow R$ be a continuous function the diagram of which in an arbitrary interval $[(n-1)l,nl]$, $n$=1,...,$N$, is shown in Fig. 3. This function will be referred to as *the fluctuation shape function* and represents a certain generalization of the saw-like function, well known in modelling of periodic laminates, [4].

**Fig. 3. A diagram of the fluctuation shape function in an arbitrary interval [(*n*-1)*l*, *nl*], *n*=1,..,*N***




We begin the modelling procedure with two assumptions. The first is the statement that for every FGL mean volume fractions are slowly varying, i. e. they satisfy conditions $\nu'(\cdot)\in SV_{\varepsilon}(l)$, $\nu''(\cdot)\in SV_{\varepsilon}(l)$. The second assumption is the formal statement that for every slowly varying function $F\in SV_{\varepsilon}(l)$ terms $O(\varepsilon F)$ can be neglected as small when compared to $F$. This assumption will be referred to as the tolerance approximation.

The subsequent considerations will be restricted to problems in which distribution of displacements across the thickness of every lamina can be approximated (with a certain tolerance $\varepsilon$ ) by a linear function. Let us denote by $\mathbf{w}(\mathbf{x},z,t)$, $\mathbf{x}=(x_1,x_2)\in\overline{\Omega}$, $z\in[0,L]$ the displacement field at time $t$. Recalling the concept of the fluctuation shape function and that of the slowly-varying function, the aforementioned restriction can be assumed in the form

$$\mathbf{w}(\mathbf{x},z,t)=\mathbf{u}(\mathbf{x},z,t)+g(z)\mathbf{v}(\mathbf{x},z,t) \tag{3}$$

where $\mathbf{u}$, $\mathbf{v}$ are assumed to be slowly varying functions of argument $z=x_3$ together with all partial derivatives which occur in the subsequent relations:

$$\mathbf{u}(\mathbf{x},\cdot,t)\in SV_{\varepsilon}(l),\ \mathbf{v}(\mathbf{x},\cdot,t)\in SV_{\varepsilon}(l) \tag{4}$$

Using the tolerance approximation we obtain

$$\mathbf{u}(\mathbf{x},z,t)=\langle\mathbf{w}\rangle(\mathbf{x},z,t)$$

for every $(\mathbf{x},z)\in\Omega\times[l/2,L-l/2]$ and every time $t$. It can be seen that $\mathbf{u}$ coincides with the averaged displacement. Hence the difference $g\mathbf{v}=\mathbf{w}-\mathbf{u}$ represents fluctuations of displacements and $\mathbf{v}$ will be called the displacement fluctuation amplitude.

Conditions (3), (4) constitute the kinematic assumption which introduces averaged displacement **u** and fluctuation amplitude **v** as the basic kinematic unknowns.

## MODEL EQUATIONS

Governing equations for kinematic unknowns **u**, **v** will be derived from the principle of stationary action. To this end the integrand in the action functional will be assumed in the form

$$\Lambda=\frac{1}{2}\langle\rho\dot{\mathbf{w}}\cdot\dot{\mathbf{w}}\rangle-\frac{1}{2}\langle\nabla\mathbf{w}:((1-\eta)\mathrm{X}+\eta\overline{\mathrm{X}}):\nabla\mathbf{w}\rangle$$

where the displacement field **w** is restricted by conditions (3), (4). It can be shown that

$$\begin{aligned}\langle\rho\rangle&=v'(z)\rho'+v''(z)\rho''\\ \langle\mathrm{X}\rangle&=v'(z)\mathrm{X}'+v''(z)\mathrm{X}''\\ \langle\overline{\mathrm{X}}\rangle&=v'(z)\overline{\mathrm{X}}'+v''(z)\overline{\mathrm{X}}''\end{aligned} \tag{5}$$

We also introduce denotations

$$\begin{aligned}[\mathrm{X}]&\equiv 2\sqrt{3}v(z)(\mathrm{X}'-\mathrm{X}'')\cdot\mathbf{e}\\ [\mathrm{X}]^{T}&\equiv 2\sqrt{3}v(z)\mathbf{e}\cdot(\mathrm{X}'-\mathrm{X}'')\\ \{\mathbf{C}\}&\equiv 12\mathbf{e}\cdot(\mathrm{X}'v''(z)+\mathrm{X}''v'(z))\cdot\mathbf{e}\end{aligned} \tag{6}$$

Using the tolerance approximation and recalling that $\mathbf{i}_3=(0,0,1)$ we shall approximate $\nabla\mathbf{w}$ by $\nabla\mathbf{u}+g'(x_3)\mathbf{i}_3\otimes\mathbf{v}+g(x_3)\overline{\nabla}\mathbf{v}$. Hence function $\Lambda$ depends on $\dot{\mathbf{u}}$, $\dot{\mathbf{v}}$, $\nabla\mathbf{u}$, $\overline{\nabla}\mathbf{v}$, **v.** The pertinent Euler-Lagrange equations can be written in the form of the equations of motion

$$\begin{aligned}&\langle\rho\rangle\ddot{\mathbf{u}}-\nabla\cdot\mathbf{S}=\mathbf{0}\\ &l^2v^2\langle\rho\rangle\ddot{\mathbf{v}}-l^2v^2\overline{\nabla}\cdot(\langle\mathrm{X}\rangle\cdot\overline{\nabla}\mathbf{v})+\mathbf{h}=\mathbf{0}\end{aligned} \tag{7}$$

and the constitutive equations

$$\begin{aligned}\mathbf{S}&=((1-\eta)\langle\mathrm{X}\rangle+\eta\langle\overline{\mathrm{X}}\rangle):\nabla\mathbf{u}+(1-\eta)[\mathrm{X}]\cdot\mathbf{v}\\ \mathbf{h}&=(1-\eta)\{\mathbf{C}\}\cdot\mathbf{v}+(1-\eta)[\mathrm{X}]^{T}:\nabla\mathbf{u}\end{aligned} \tag{8}$$

By virtue of (2) the mean density $\eta$ of open microdefects has to satisfy condition

$$\eta\in\partial\varphi(\mathbf{i}_3\cdot\nabla\mathbf{u}\cdot\mathbf{i}_3) \tag{9}$$

Equations of motion (7) together with constitutive equations (8), condition (9) and natural boundary conditions $\mathbf{S}\cdot\mathbf{n}=\mathbf{t}$, where **t** are boundary tractions, represent a continuum model of the functionally graded laminate with a continuous distribution of microdefects on the interfaces between laminae. It has to be emphasized that all coefficients in equations (7) are slowly varying function of $z=x_3$. For a laminated medium with a periodic structure all coefficients in the above model equations are constant and the obtained results reduce to those discussed in [3]. The characteristic feature of equations (7) is that they describe the effect of the layer thickness *l* on the overall (macroscopic) behavior of the functionally graded laminate under consideration.

Now let us assume that in equations (7) terms involving the layer thickness *l* are neglected. In this case $\mathbf{h}=\mathbf{0}$ and we obtain

$$\mathbf{v}=-\{\mathbf{C}\}^{-1}\cdot[\mathrm{X}]^{T}:\nabla\mathbf{u} \tag{10}$$

where $\{\mathbf{C}\}^{-1}$ is the inverse to the non-singular linear transformation $\{\mathbf{C}\}$. Introducing the following tensor of effective (homogenized) elastic moduli

$$\mathrm{X}^{h}=(1-\eta)\left(\langle\mathrm{X}\rangle-[\mathrm{X}]\cdot\{\mathbf{C}\}^{-1}\cdot[\mathrm{X}]^{T}\right)+\eta\langle\overline{\mathrm{X}}\rangle \tag{11}$$

we arrive at equation

$$\langle\rho\rangle\ddot{\mathbf{u}}-\nabla\cdot\left(\mathrm{X}^{h}:\nabla\mathbf{u}\right)=\mathbf{0} \tag{12}$$

Equations (10), (12) together with condition (9) represent the asymptotic approximation of the general model equations derived in this section.

## CONCLUSIONS

The following conclusions and remarks summarize the new results obtained in this note:

1. A macroscopic (averaged) model of a functionally graded laminates (FGL) with a continuous distribution of microdefects on the interfaces between adjacent laminae was derived.
2. The considerations were carried out in the framework of the linear elasticity theory but the resulting model equations are nonlinear due to the existence of the free boundary separating regions with open and closed microdefects.
3. The difference between results obtained in this note and those derived in [5] is due to the uncorrected form of the fluctuation shape function $g(\cdot)$ introduced in [5].

An extended version of this note including also an example of application of the proposed model will be given in the separate paper.

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