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ON A CERTAIN METHOD OF STIFFENED PLATES MODELING

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ABSTRACT

A method of stiffened plates modeling is being described in the paper. The method consists on dividing a construction into plate elements and stiffeners. Each of these elements is calculated independently, taking unknown forces of co-operation between these elements into consideration. These forces are determined from the conditions of the continuity of displacements on the common surfaces of the plate elements and stiffeners. The plates under consideration are made of ferroconcrete and the stiffeners are on the one side of the plate.

Key words: plates, stiffened plates, nonhomogeneous plates, modeling of plates.

INTRODUCTION

The plates are being reinforced by stiffeners in order to increase their load capacity. The examples of stiffened-plate-type structures are slab floors and bridge plates. If the joint between a plate and a stiffener is continuous, such plates are called stiffened (reinforced by stiffeners). Stiffened plates are nonhomogeneous and anisotropic. Various methods of their modeling may be found in the literature. In 1914, introducing the notion of equivalent stiffness, M. T. Huber worked out the model of a constructionally ortotropic plate and applied it to the calculations of ferroconcrete plates [1-2]. The method of equivalent stiffness (modules) can be also applied to stiffened plates [3-6]. The equivalent stiffness can be calculated by means of many ways, e.g. direct methods [6], method of microlocal parameters [7-8], method of asymptotic homogenization [9-12] or averaging tolerance technique (nonasymptotic homogenization) [13-15]. The other approach is presented in [16-17], where the stiffened plates are described as the homogeneous plates with variable stiffness or variable thickness [18]. The solution of problems of mechanics, dynamics as well as stability of rectangular isotropic plates reinforced by stiffeners was presented in 1954 [19-20]. The monograph [21] is dedicated to static, dynamic and stability analysis of stiffened plates; it also consists the extensive literature on this topic.

Thin plates stiffened on the one side (Fig. 1) are being considered in presented paper. To model such plates, an analytic method has been applied, which consists on dividing a construction into plate elements and stiffeners.

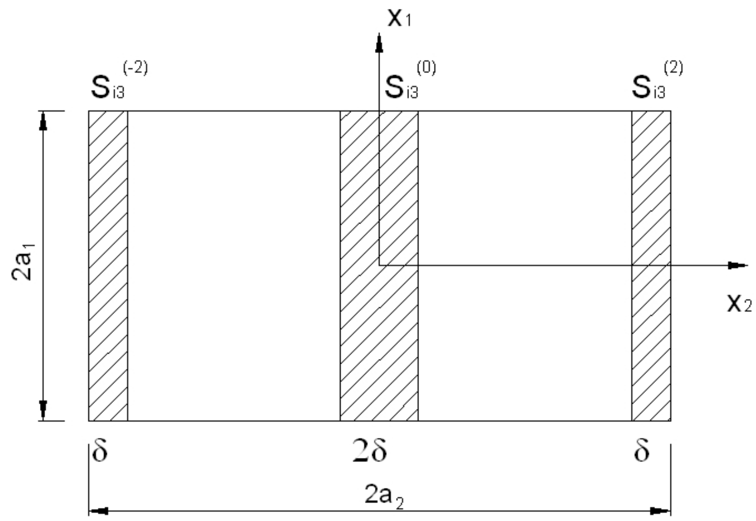
Every of these parts is being calculated independently, taking an unknown co-operation between them into consideration. This co-operation is determined from the conditions of the continuity of displacements on the common surfaces of the plate elements and stiffeners.

THE MODELING OF DISPLACEMENT FIELD IN A PLATE AND STIFFENERS

One shall consider a thin rectangular isotropic plate, reinforced on the bottom by the series of thin isotropic stiffeners, distributed parallel to each other.

The size of a typical fragment of the plate in a medium surface by $2a_j$ (j = 1, 2) may be denoted. The thickness of the plate and the stiffeners shall be denoted by h_1 and h_2, respectively. The widths of the outermost stiffeners are assumed as the same and equal δ, whereas the width of the central stiffener is assumed as 2δ. It may be also assumed that the plate is supported at the stiffeners or mounted at the edges. On the upper surface of the plate an arbitrary load works.

Fig. 1. The scheme of a stiffened plate



The origin of Carthesian coordinate system $Ox_1x_2x_3$ is assumed to be hold in the geometric center of the plate, the axis Ox_1 is directed along the stiffeners, the axis Ox_3 – perpendicularly to it (Fig. 1).

It may be assumed that the horizontal displacement field is of a form:

$$u_1(x_1, x_2 x_3) = -x_3 w,_1(x_1, x_2) - U_1(x_1, x_2); \; u_2(x_{1,} x_2, x_3) = -x_3 w,_2(x_{1,} x_2) \tag{1}$$

where $x_3 \in \left(-\frac{h_1}{2}, \frac{h_1}{2}\right)$, w is the deflection of the plate, U_1 – an unknown function which may be determined later from the conditions of the continuity of displacements.

In a similar way it may be assumed that the horizontal displacements of the stiffener № i (i = 0, 2, Fig. 1) are equal

$$u_1^{(i)}(x_1, x_3) = -x_3 v,_1^{(i)}(x_1) - V_1^{(i)}(x_1) \tag{2}$$

In (2), the function $v^{(i)}(x_i)$ is the deflection of the stiffener № i, whereas $V^{(i)}$ – a correcting function describing the horizontal displacement of the stiffener № i.
On the surfaces of contact between the plate and the stiffeners, the conditions of the continuity of deflection $w(x_1, x_2)_{|x_2=d_i} = v^{(i)}(x_1)$ (i = 0, 2) as well as of the horizontal displacements $u_1\big|_{x_2=d_i} = u_1^{(i)}$ are to be satisfied; d_i denotes the distance of the stiffener № i from the central axis of the plate. In the case i = 0, $\delta_0 = 0$ denotes the position of the central stiffener and $\delta_2 = a_2$ corresponds to the outermost stiffeners.

Let $w^{(i)}(x_1)$ denote the deflection of the plate in the section corresponding to the stiffener № i, $w^{(i)}(x_1)= w\big|_{x_2=d_i}$, whereas $v^{(i)}(x_1)$ – the deflection of this very stiffener. If the condition of the continuity of deflection of the plate and the stiffener $w^{(i)}(x_1)=v^{(i)}(x_1)$ is fulfilled, then the condition of the angle continuity with normal rotation $w,_1^{(i)}=v,_1^{(i)}$ is fulfilled automatically. To satisfy the condition of the continuity of horizontal deflection, it is sufficient to satisfy the condition $U_1(x_1,x_2)_{|x_2=d_i}=V_1^{(i)}(x_1)$. In order to do this, one defines the function $U_1(x_1, x_2)$ as follows:

$$U_1(x_1,x_2)=\begin{cases} V_1^{(-2)}(x_1), & dla\ -a_2 \le x_2 \le -(a_2-\delta) \\ U_1^{(-1)}(x_1), & dla\ -(a_2-\delta)\le x_2 \le -\delta \\ V_1^{(0)}(x_1), & dla\ -\delta \le x_2 \le \delta \\ U_1^{(1)}(x_1), & dla\ \delta \le x_2 \le (a_2-\delta) \\ V_1^{(2)}(x_1), & dla\ a_2-\delta \le x_2 \le a_2. \end{cases} \tag{3}$$

where $U_1^{(i)}(x_1)$ are the functions of the horizontal displacement of non-stiffened parts of the plate. As follows from (3), the function $U_1(x_1, x_2)$ is discontinuous. One may expand it into a Fourier series:

$$U_1(x_1,x_2)=\sum_{n=1}^{\infty} P_n(x_1)\cos\delta_n^{[2]}x_2 \tag{4}$$

where $\delta_n^{[2]}=\dfrac{(2n-1)\pi}{2a_2}$. The coefficients $P_n(x_1)$ of the series (4) are of a form:

$$P_n(x_1)=\frac{1}{a_2}\left[\int_{-a_2}^{-(a_2-\delta)} V_1^{(-2)}(x_1)\cos\delta_n^{[2]}x_2dx_2+\int_{-(a_2-\delta)}^{-\delta} U_1^{(-1)}(x_1)\cos\delta_n^{[2]}x_2dx_2+\right.$$
$$\left.+\int_{-\delta}^{\delta} V_1^{(0)}(x_1)\cos\delta_n^{[2]}x_2dx_2+\int_{\delta}^{a_2-\delta} U_1^{(1)}(x_1)\cos\delta_n^{[2]}x_2dx_2+\int_{a_2-\delta}^{a_2} V_1^{(2)}(x_1)\cos\delta_n^{[2]}x_2dx_2\right] \tag{5}$$

It may be assumed that a load operating at the plate and the boundary conditions are such that $V_1^{(-2)}(x_1)=V_1^{(2)}(x_1)$, $U_1^{(-1)}(x_1)=U_1^{(1)}(x_1)$ – then the formula (5) substantially simplifies itself. Since the functions $U_1^{(i)}(x_1)$ and $V_1^{(j)}(x_1)$ do not depend on the variable x_2, then the formula (5) may be written in a form:

$$P_n(x_1)=C_{1(n)}^{(0)}V_1^{(0)}(x_1)+C_{1(n)}^{(1)}U_1^{(1)}(x_1)+C_{1(n)}^{(2)}V_1^{(2)}(x_1), \tag{6}$$

where the following denotations have been introduced:

$$C_{1(n)}^{(0)}=\frac{2\left[\sin\delta_n^{[2]}\delta\right]}{a_2\delta_n^{[2]}},\ C_{1(n)}^{(1)}=\frac{2\left[\sin\delta_n^{[2]}(a_2-\delta)-\sin\delta_n^{[2]}\delta\right]}{a_2\delta_n^{[2]}},\ C_{1(n)}^{(2)}=\frac{2\left[\sin\delta_n^{[2]}a_2-\sin\delta_n^{[2]}(a_2-\delta)\right]}{a_2\delta_n^{[2]}}. \tag{7}$$

In that case the function $U_1(x_1, x_2)$ assumes a form:

$$U_1(x_1,x_2)=\sum_{n=1}^{\infty}\left[C_{1(n)}^{(0)}V_1^{(0)}(x_1)+C_{1(n)}^{(1)}U_1^{(1)}(x_1)+C_{1(n)}^{(2)}V_1^{(2)}(x_1)\right]\cos\delta_n^{[2]}x_2\,. \tag{8}$$

It is easy to check out that every discontinuous static or kinematic quantity evoked by existing of the stiffeners may be presented in a similar way.

DEFINITION OF A STRESS STATE IN THE PLATE AND THE STIFFENERS

The stress state is determined separately for the plate and the stiffeners. The normal (σ_{11}, σ_{22}) and tangential (σ_{12}) stresses in the plate are described by the physical equations of plane stress:

$$\sigma_{11}=\frac{E}{1-\nu^2}\left(\varepsilon_{11}+\nu\varepsilon_{22}\right),\quad \sigma_{22}=\frac{E}{1-\nu^2}\left(\varepsilon_{22}+\nu\varepsilon_{11}\right),\quad \sigma_{12}=G\gamma_{12} \tag{9}$$

where E and ν are Young modulus and Poisson's coefficient of the plate, respectively. Substituting the displacements (1) to the deformations, these whereas – to the stresses (9), one obtains:

$$\sigma_{11}=-\frac{E}{1-\nu^2}\left[x_3\left(w_{,11}+\nu w_{,22}\right)+U_{1,1}\right];\quad \sigma_{22}=-\frac{E}{1-\nu^2}\left[x_3\left(w_{,22}+\nu w_{,11}\right)+\nu U_{1,1}\right];$$

$$\sigma_{12}=-\frac{E}{2(1+\nu)}\left(2x_3 w_{,12}+U_{1,2}\right). \tag{10}$$

According to the Kirchhoff's theory, the stresses σ_{13}, σ_{23} are equal

$$\begin{aligned}\sigma_{13}&=-\int\sigma_{11,1}dx_3-\int\sigma_{12,2}dx_3+f_1(x_1,x_2)\\ \sigma_{23}&=-\int\sigma_{21,1}dx_3-\int\sigma_{22,2}dx_3+f_2(x_1,x_2)\end{aligned} \tag{11}$$

where – for the sake of simplicity – the body forces have been neglected. The functions $f_\alpha(x_1, x_2)$, $\alpha = 1, 2$ existing in (11) are arbitrary integration constants. The derivatives of the normal and tangential stresses in the integrands may be obtained by the differentiation of the physical equations (10) on account of the variables x_1, x_2. Then, integrating them on account of the variable x_3, one obtains the formulas for the stresses σ_{13}, σ_{23} in the plate:

$$\begin{aligned}\sigma_{13}&=\frac{E}{\left(1-\nu^2\right)}\left[\frac{x_3^2}{2}\nabla^2 w_{,1}+x_3\Psi_1(x_1,x_2)\right]+f_1(x_1,x_2)\\ \sigma_{23}&=\frac{Ex_3^2}{2\left(1-\nu^2\right)}\nabla^2 w_{,2}+f_2(x_1,x_2)\end{aligned} \tag{12}$$

where the denotation has been introduced:

$$\Psi_1=U_{1,11}+\frac{(1-\nu)}{2}U_{1,22}. \tag{13}$$

From the conditions that the stresses (12) on the upper plate surface must be equal to 0

$$\sigma_{13}\Big|_{x_1=-\frac{h_1}{2}}=0,\qquad \sigma_{23}\Big|_{x_3=-\frac{h_1}{2}}=0 \tag{14}$$

one determines the functions $f_\alpha(x_1, x_2)$, which – after substituting to (12) – yield

$$\begin{aligned}\sigma_{13}&=\frac{E}{1-\nu^2}\left[\frac{1}{2}\left(x_3^2-\frac{h_1^2}{4}\right)\nabla^2 w_{,1}+\left(x_3+\frac{h_1}{2}\right)\Psi_1\right];\\ \sigma_{23}&=\frac{1}{2}\left(x_3^2-\frac{h_1^2}{4}\right)\frac{E}{1-\nu^2}\nabla^2 w_{,2}\end{aligned} \tag{15}$$

The calculated stresses (15) are equal to 0 on the bottom surface of the plate everywhere but the stiffened area. In this area, the conditions of the continuity of stresses of the plate and the stiffeners must be satisfied.

It may be denoted the stress $\sigma_{13}(x_1, x_2)$ on the bottom surface of the plate by $\tau_{13}^{(*)}(x_1,x_2)$ and the stress $\sigma_{13}^{(i)}(x_1)$ on the upper surface of the plate by $\tau_{13}^{(i)}(x_1)$. One assumes

$$\tau_{13}^{(*)}(x_1,x_2)=\begin{cases}\tau_{13}^{(-2)}(x_1), & dla\ -a_2\le x_2\le-(a_2-\delta)\\ 0, & dla\ -(a_2-\delta)\le x_2\le-\delta\\ \tau_{13}^{(0)}(x_1), & dla\ -\delta\le x_2\le\delta\\ 0, & dla\ \delta\le x_2\le(a_2-\delta)\\ \tau_{13}^{(2)}(x_1), & dla\ a_2-\delta\le x_2\le a_2.\end{cases} \tag{16}$$

On the bottom and upper surface of the plate, the following boundary conditions must be satisfied:

$$\sigma_{13}\Big|_{x_3=\frac{h_1}{2}}=\tau_{13}^{(*)},\qquad \sigma_{23}\Big|_{x_3=\frac{h_1}{2}}=0 \tag{17}$$

To fulfill the condition (17), one changes the discontinuous function (16) of the stress $\tau_{13}^{(*)}(x_1,x_2)$ into the continuous function $\tau_{13}(x_1,x_2)$, using the procedure described in Section 1 (this function may be expanded into the Fourier's series on account of the variable x_2):

$$\tau_{13}^{(*)}(x_1,x_2)\sim\tau_{13}(x_1,x_2)=\sum_{n=1}^{\infty}T_n(x_1)\cos\delta_n^{[2]}x_2 \tag{18}$$

The coefficients of this series have a form:

$$T_n(x_1)=\frac{2}{a_2}\left[\tau_{13}^{(0)}(x_1)\int_{-\delta}^{\delta}\cos\delta_n^{[2]}x_2\right]+\frac{1}{a_2}\left[\tau_{13}^{(2)}(x_1)\int_{a_2-\delta}^{a_2}\cos\delta_n^{[2]}x_2\right]=C_{1(n)}^{(0)}\tau_{13}^{(0)}(x_1)+C_{1(n)}^{(2)}\tau_{13}^{(2)}(x_1). \tag{19}$$

The stress $\tau_{13}^{(i)}(x_1)$ existing in (19) may be defined as the limit stress $\sigma_{13}^{(i)}(x_1)$ on the upper surface of the stiffener № *i*:

$$\tau_{13}^{(i)}(x_1)=\sigma_{13}^{(i)}(x_1,x_3)\Big|_{x_3=-h_2/2} \tag{20}$$

Subsequently, the stress $\sigma_{13}^{(i)}(x_1,x_3)$ is determined from the equation of equilibrium and physical equations written for the stiffener № *i*:

$$\sigma_{13}^{(i)}=-\int\sigma_{11,1}^{(i)}dx_3+f_1^{(i)}(x_1)=E_i\left[\frac{x_3^2}{2}v_{,111}^{(i)}+x_3V_{1,11}^{(i)}\right]+f_1^{(i)}(x_1) \tag{21}$$

This stress must be equal to 0 on the bottom surface of the stiffener:

$$\sigma_{13}^{(i)}\Big|_{x_3=\frac{h_2}{2}}=E_i\left[\frac{h_2^2}{8}v_{,111}^{(i)}+\frac{h_2}{2}V_{1,11}^{(i)}\right]+f_1^{(i)}(x_1)=0 \tag{22}$$

what is ensured by an appropriate choice of the function $f_1^{(i)}(x_1)$. Then the stress $\sigma_{13}^{(i)}$ assumes a form:

$$\sigma_{13}^{(i)}=E_i\left[\frac{1}{2}\left(x_3^2-\frac{h_2^2}{4}\right)v_{,111}^{(i)}+\left(x_3-\frac{h_2}{2}\right)V_{1,11}^{(i)}\right] \tag{23}$$

One shall define the stress on the upper surface of the stiffener:

$$\sigma_{13}^{(i)}\Big|_{x_3=-\frac{h_2}{2}}=\tau_{13}^{(i)}=E_i\left[\frac{1}{2}\left(x_3^2-\frac{h_2^2}{4}\right)v_{,111}^{(i)}+\left(x_3-\frac{h_2}{2}\right)V_{1,11}^{(i)}\right]_{x_3=-\frac{h_2}{2}}=-E_ih_2V_{1,11}^{(i)}. \tag{24}$$

Knowing $\tau_{13}^{(i)}$ as well as using (18) and (19), one constructed stress function $\tau_{13}(x_1,x_2)$ in a form:

$$\tau_{13}(x_1,x_2) = -h_2\sum_{n=1}^{\infty}\left[C_{1(n)}^{(0)}E_0V_{1,11}^{(0)}(x_1) + C_{1(n)}^{(2)}E_2V_{1,11}^{(2)}(x_1)\right]\cos\delta_n^{[2]}x_2 \tag{25}$$

One shall write down the stress $\sigma_{13}(x_1,x_2,x_3)$ on the bottom surface of the plate in a form:

$$\sigma_{13}\Big|_{x_3=\frac{h_1}{2}} = \frac{E}{1-\nu^2}h_{1\Psi_1} = \frac{E}{1-\nu^2}h_1\left(U_{1,11} + \frac{(1-\nu)}{2}U_{1,22}\right). \tag{26}$$

Using (8), the derivatives of the function $U_1(x_1, x_2)$ may be determined and the following formula of the function $\Psi_1(x_1, x_2)$ may be obtained:

$$\Psi_1 = \sum_{n=1}^{\infty}\left\{C_{1(n)}^{(0)}\left[V_{1,11}^{(0)} - \frac{(1-\nu)}{2}\delta_n^{[2]^2}V_1^{(0)}\right] + C_{1(n)}^{(1)}\left[U_{1,11}^{(1)} - \frac{1-\nu}{2}\delta_n^{[2]^2}U_1^{(0)}\right] + \right.$$
$$\left. + C_{1(n)}^{(2)}\left[V_1{,}_{11}^{(2)} - \frac{(1-\nu)}{2}\delta_n^{[2]^2}V_1^{(2)}\right]\right\}\cos\delta_n^{[2]}x_2 \tag{27}$$

From the condition of the continuity of stresses σ_{13}, σ_{23} on the common surface of the plate and the stiffeners one obtains differential relations between the function of the horizontal displacement of the plate U_1 and the stiffeners $V_1^{(i)}$ and their derivatives:

$$V_{1(k),11}^{(0)} - K_{0(k)}V_{1(k)}^{(0)} = 0;\quad U_{1(k),11}^{(1)} - K_{1(k)}U_{1(k)}^{(1)} = 0;\quad V_{1(k),11}^{(2)} - K_{2(k)}V_{1(k)}^{(2)} = 0 \tag{28}$$

where the denotations have been introduced:

$$K_{0(k)} = \frac{Eh_1(1-\nu)\delta_k^{[2]^2}}{2\left[h_2(1-\nu^2)E_0 + Eh_1\right]},\quad K_{1(k)} = \frac{(1-\nu)\delta_k^{[2]^2}}{2},\quad K_{2(k)} = \frac{Eh_1(1-\nu)\delta_k^{[2]^2}}{2\left[h_2(1-\nu^2)E_2 + Eh_1\right]} \tag{29}$$

For the case of a symmetrical construction the solution of this system of equations comes to a form:

$$V_{1(k)}^{(0)}(x_1) = L_{1(k)}^{(0)}\sinh\left(\sqrt{K_{0(k)}x_1}\right),\ U_{1(k)}^{(1)}(x_1) = L_{1(k)}^{(1)}\sinh\left(\sqrt{K_{1(k)}x_1}\right),\ V_{1(k)}^{(2)}(x_1) = L_{1(k)}^{(2)}\sinh\left(\sqrt{K_{2(k)}x_1}\right) \tag{30}$$

The solution (30) is being used to define the following functions:

$$V_1^{(0)}(x_1) = \sum_{k-1}^{N}V_{1(k)}^{(0)}(x_1) = \sum_{k-1}^{N}L_{1(k)}^{(0)}\sinh\left(\sqrt{K_{0(k)x_1}}\right),$$
$$U_1^{(1)}(x_1) = \sum_{k=1}^{N}U_{1(k)}^{(1)}(x_1) = \sum_{k=1}^{N}L_{1(k)}^{(1)}\sinh\left(\sqrt{K_{1(k)}x_1}\right), \tag{31}$$
$$V_1^{(2)}(x_1) = \sum_{k=1}^{N}V_{1(k)}^{(2)}(x_1) = \sum_{k=1}^{N}L_{1(k)}^{(2)}\sinh\left(\sqrt{K_{2(k)}x_1}\right)$$

Replacing (27) by (31), one obtains

$$\Psi_1(x_{1,}x_2) = \sum_{k=1}^{N}\sum_{n=1}^{N}\left\{N_{1n(k)}^{(0)}\sinh\sqrt{K_{0(k)}x_1} + N_{1n(k)}^{(1)}\sinh\sqrt{K_{1(k)}x_1} + N_{1n(k)}^{(2)}\sinh\sqrt{K_{2(k)}x_1}\right\}\cos\delta_n^{(2)}x_2) \tag{32}$$

where the denotation: $N_{1n(k)}^{(i)} = C_{1(n)}^{(i)}L_{1(k)}^{(i)}\left[K_{i(n)} - \frac{(1-\nu)}{2}\delta\right]$ has been introduced.

Now one shall use the conditions of the continuity of deflections and normal stresses on the surfaces of contact between the plate and the stiffeners. The normal stress σ_{33} is determined from the equation of equilibrium:

$$\sigma_{33} = -\frac{E}{\left(1-\nu^2\right)}\left[\frac{1}{2}\left(\frac{x_3^3}{3} - \frac{h_1^2 x_3}{4}\right)\nabla^2\nabla^2 w + \left(\frac{x_3^2}{2} + \frac{h_1}{2}x_3\right)\Psi_{1,1}\right] + f_3\left(x_1, x_2\right) \tag{33}$$

Denoting the limit stress $\sigma_{33}(x_1, x_2)$ on the bottom surface of the plate by $S_{33}(x_1, x_2)$:

$$\sigma_{33}\Big|_{x_3=\frac{h_1}{2}} = \frac{Eh^3}{24\left(1-\nu^2\right)}\nabla^2\nabla^2 w - \frac{3Eh_1^2}{8\left(1-\nu^2\right)}\Psi_{1,1} + f_3\left(x_1, x_2\right) = -S_{33} \tag{34}$$

one may present the function $S_{33}(x_1, x_2)$ in the form of a trigonometric series:

$$S_{33}\left(x_1, x_2\right) = \sum_{n=1}^{\infty}\left[C_{1(n)}^{(0)} S_{13}^{(0)}\left(x_1\right) + C_{1(n)}^{(2)} S_{13}^{(2)}\left(x_1\right)\right]\cos\delta_n^{[2]} x_2 \tag{35}$$

where $S_{33}^{[i]}\left(x_1, x_2\right)$ is the normal stress on the upper surface of the stiffeners $\sigma_{33}^{[i]}\left(x_1, x_2\right)$:

$$S_{33}^{(i)}\left(x_1\right) = \sigma_{33}^{(i)}{}_{x_3=-\frac{h_2}{2}} \tag{36}$$

If the stress σ_{33} has to fulfill the condition (34), it assumes a form:

$$\sigma_{33} = -\frac{E}{\left(1-\nu^2\right)}\left[\frac{1}{2}\left(\frac{x_3^3}{3} - \frac{h_1^2 x_3}{4} + \frac{h_1^3}{12}\right)\nabla^2\nabla^2 w + \left(\frac{x_3^2}{2} + \frac{x_3 h_1}{2} - \frac{3h_1^2}{8}\right)\Psi_{1,1}\right] - S_{33} \tag{37}$$

Subsequently, from the condition on the upper surface of the plate $x_3 = -\dfrac{h_1}{2}$ one has:

$$\sigma_{33} = -\frac{E}{\left(1-\nu^2\right)}\left[\frac{1}{2}\left(-\frac{h_1^3}{24} + \frac{h_1^3}{8} + \frac{h_1^3}{12}\right)\nabla^2\nabla^2 w - \frac{h_1^2}{2}\Psi_{1,1}\right] - S_{33} = -q \tag{38}$$

Comparing (37) and (38), the equation of bending of thin isotropic plates with consideration of the influence of shield forces may be obtained:

$$\nabla^2\nabla^2 w = \frac{q - S_{33}}{D} + \frac{6}{h_1}\Psi_{1,1}; \tag{39}$$

where $D = \dfrac{Eh^3}{12\left(1-\nu^2\right)}$ is the known stiffness of a plate on account of bending, q – an external load applied to an upper surface of a plate.

In the equation (39) an unknown function $S_{33}(x_1, x_2)$ exists, which shall be determined from the comparison to the limit normal stress $\sigma_{33}^{(i)}\left(x_1\right)$ on the upper surface of the stiffener № i; this stress is defined by the relation (36). At first, one determines the stress $\sigma_{33}^{(i)}\left(x_1\right)$ from the equation of equilibrium (similarly as for the plate):

$$\sigma_{33}^{(i)} = -E_i\left[\frac{1}{2}\left(\frac{x_3^3}{3} - \frac{h_2^2 x_3}{4} + \frac{h_2^3}{12}\right)v_{,1111}^{(i)} + \left(\frac{x_3^2}{2} - \frac{x_3 h_2}{2} - \frac{h_2^2}{8}\right)\right]V_{1,111}^{(i)}. \tag{40}$$

Then, one satisfies the boundary condition on the upper surface of the stiffener

$$\sigma_{33}^{(i)}\Big|_{x_3=-\frac{h_2}{2}} = -S_{33}^{(i)}$$

and the basic equations of bending of the stiffener (bar) considering the shield forces may be obtained:

$$v_{,1111}^{(i)} + \frac{6}{h_2} V_{1,111}^{(i)} = \frac{S_{33}^{(i)}}{D_i}, \tag{41}$$

where $D_i = \frac{E_i h_2^{(i)3}}{12}$ is the bending rigidity of the stiffener № i. The stress $S_{33}^{(i)}$, which is treated as the load applied to the upper surface of the stiffener № i, is unknown, as well as the deflection of this stiffener. To solve the equation (41), one presents the stress $S_{33}^{(i)}(x_1)$ in a form of a trigonometric series:

$$S_{33}^{(i)}(x_1) = \sum_{m=1}^{\infty} Z_m^{(i)} \cos \delta_m^{(i)} x_1 \tag{42}$$

where $\delta_m^{(1)} = \frac{(2m-1)\pi}{2a_1}$, while the coefficients $Z_m^{(i)}$ are unknown constant parameters. Replacing (41) by (42), one obtains the ordinary nonhomogeneous differential equation of 4th order for an unknown function $v^{(i)}$ of the deflection of the stiffener № i:

$$v_{,1111}^{(i)} = \frac{1}{D_i} \sum_{m=1}^{\infty} Z_m^{(i)} \cos \delta_m^{[1]} x_1 - \frac{6}{h_2} \sum_{k=1}^{N} L_{1(k)}^{(i)} \left(K_{i(k)}\right)^{3/2} \cosh\left(\sqrt{K_{i(k)}} x_1\right) \tag{43}$$

In a similar way the equations (39) may be transformed. Using the series (42) and the form (35), the function $S_{33}(x_1)$ may be given in a form of a double trigonometric series:

$$S_{33}(x_1) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i=0}^{2} \left[C_{1(n)}^{(i)} Z_m^{(i)}\right] \cos \delta_m^{[1]} x_1 \cos \delta_n^{[2]} x_2 \tag{44}$$

Analogically one presents the external load:

$$q(x_1, x_2) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{mn} \cos \delta_m^{[1]} x_1 \cos \delta_n^{[2]} x_2 ; \tag{45}$$

where q_{mn} are the coefficient of the series.

Using (32), one determinates the derivatives of Ψ_1, which − after substitution to (39) − come to the following equation:

$$\nabla^2 \nabla^2 w = \frac{1}{D} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{mn} \cos \delta_m^{[1]} x_1 \cos \delta_n^{[2]} x_2 - \frac{1}{D} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[C_{1(n)}^{(0)} Z_m^{(0)} + C_{1(n)}^{(2)} Z_m^{(2)}\right] \cos \delta_m^{[1]} x_1 \cos \delta_n^{[2]} x_2 +$$

$$+ \frac{6}{h_1} \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{i=0}^{2} \left\{N_{1n(k)}^{(i)} \sqrt{K_{i(k)}} \cosh \sqrt{K_{i(k)}} x_1 \right\} \cos \delta_n^{[2]} x_2 \tag{46}$$

This is a nonhomogeneous differential equation of 4th order on account of an unknown function of the plate deflection w.

THE SOLUTION OF THE BASIC EQUATIONS

One seeks the general solutions of the differential equations (43, 46) in a form of the sum of the general integral of a homogeneous equation and the particular integral of a nonhomogeneous equation. The general solution of (46) has a form:

$$w_0(x_1,x_2)=\sum_{k=1}^{\infty}\sum_{\nu=1}^{4}R_{\nu(k)}W_{\nu(k)}(x_1,x_2) \tag{47}$$

where $W_{\nu(k)}(x_1, x_2)$ are the functions of the plate deflection [22]. The unknown coefficients $R_{\nu(k)}$ are defined from the boundary conditions of the plate. The particular solution of this equation has a form:

$$w_*=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos\delta_m^{[1]}x_1\cos\delta_n^{[2]}x_2+\sum_{n=1}^{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\left\{S_{1n(k)}^{(i)}\cosh\sqrt{K_{i(k)}}x_1\cos\delta_n^{[2]}x_2\right\}. \tag{48}$$

The general solution of (43) shall be written in a form:

$$v=C_0^{(i)}+C_2^{(i)}x_1^2+\sum_{m=1}^{\infty}D_m^{(i)}\cos\delta_m^{[1]}x_1+\sum_{k=1}^{N}F_{1(k)}^{(i)}\cosh\left(\sqrt{K_{i(k)}x_1}\right) \tag{49}$$

In (49) one knows only the coefficients q_{mn}, while all of the rest are unknown. The coefficients existing in (48), (49) are determined from the conditions of the continuity of deflections on the surfaces of contact between the plate and the stiffener.

One shall differentiate the particular integrals (48), (49) of the equations (43), (46):

$$\begin{aligned}\nabla^2\nabla^2 w=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\left(\delta_m^{[1]^2}+\delta_n^{[1]^2}\right)^2\cos\delta_m^{[1]}x_1\cos\delta_n^{[2]}x_2+\\&+\sum_{n=1}^{N}\sum_{k=1}^{N}\sum_{i=0}^{2}\left\{S_{1n(k)}^{(i)}\left(K_{i(k)}-\delta_n^{[2]^2}\right)^2\cosh\sqrt{K_{i(k)}}x_1\cos\delta_n^{[2]}x_2\right\},\end{aligned} \tag{50}$$

$$v_{,1111}^{(i)}=\sum_{m=1}^{\infty}D_m^{(i)}\delta_m^{[1]^4}\cos\delta_m^{[1]}x_1+\sum_{k=1}^{N}F_{1(k)}^{(i)}K_{i(k)}^2\cos\sqrt{K_{i(k)}}x_1 \tag{51}$$

Comparing (46), (50) to (43), (51), the relations between the parameters B_{mn}, q_{mn} and $\left[C_{1(n)}^{(0)}Z_m^{(0)}+C_{1(n)}^{(2)}Z_m^{(2)}\right]$ as well as between the parameters $S_{1n(k)}^{(i)}$ and $N_{1n(k)}^{(i)}$ may be obtained:

$$\begin{aligned}B_{mn}\left(\delta_m^{[1]^2}+\delta_n^{[1]^2}\right)^2&=\frac{q_{mn}-\left[C_{1(n)}^{(0)}Z_m^{(0)}+C_{1(n)}^{(2)}Z_m^{(2)}\right]}{D}\\ S_{1n(k)}^{(i)}\left(K_{i(k)}-\delta_n^{[2]^2}\right)^2&=\frac{6}{h_2}N_{1n(k)}^{(i)}\sqrt{K_{i(k)}}\end{aligned} \tag{52}$$

$$\delta_m^{[1]^4}D_m^{(i)}=\frac{1}{D_i}Z_m^{(i)}\ ;\qquad K_{i(k)}^2F_{1(k)}^{(i)}=-\frac{6}{h_2}L_{1(k)}^{(i)}\left(K_{i(k)}\right)^{3/2} \tag{53}$$

The remaining parameters are determined from the conditions of the continuity of deflection of the plate (47) and the stiffener (48) on their common surfaces:

$$D_m^{(i)}=\sum_{n=1}^{\infty}B_{mn}\cos\delta_n^{[2]}d_i\,, \tag{54}$$

$$F_{1(k)}^{(i)}=\sum_{n=1}^{\infty}S_{1(n)k}^{(i)}\cos\delta_n^{[2]}d_i \tag{55}$$

The equation

$$\sum_{k=1}^{\infty}\sum_{\nu=1}^{4} R_{\nu(k)} W_{\nu(k)}(x_1, d_2) = c_0^{(i)} + c_2^{(i)} x_1^2 \, ; \tag{56}$$

may be fulfilled – in an approximate way – only in two points: $x_1 = 0$ and $x_1 = a_1$.

CONCLUSIONS

The new constructed model of thin isotropic rectangular plates reinforced on the bottom side by the series of stiffeners distributed parallel is presented in the paper. In this model the conditions of the continuity of stress are satisfied. The obtained system of equations has been solved analytically.

In the model, the boundary conditions at the plate edges are satisfied within a great accuracy.

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