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A CONTRIBUTION TO THE MODELLING OF PERIODICALLY LAMINATED ELASTIC SOLIDS

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ABSTRACT

In the modern civil engineering an important role play new materials like composites and laminates. In this paper the object of analysis is a behaviour of periodic, micro-laminated two-component elastic solids. The modelling question is how to describe both micro- and macro-response of linear elastic periodic laminate. In order to answer this question we propose a new model of the solid under consideration. In contrast to the higher-order homogenization model, [3,4], the model is based on some heuristic physical assumptions rather than on the formal asymptotic expansions. The main result is that under certain conditions the obtained model equations can be decomposed into equations describing behaviour of a laminate independently on a macro- and micro-level. This decomposition holds for the laminates with a weak transversal inhomogeneity which will be defined in the paper.

Key words: laminates, mathematical modelling.

INTRODUCTION

The recent literature on the response of the linear elastic laminated solids is rather extensive. The simplest mathematical model is that of the homogenized medium with material properties described by the known effective modulae [1,2]. However, the above model is unable to describe many important phenomena like dispersion and attenuation of waves or near-boundary and near-initial fluctuations of displacements, [7]. For finding solutions to the initial/boundary value problems in periodic media the higher order homogenization, based on the asymptotic expansions of the displacement field, was used as a tool of analysis, [3]. An alternative asymptotic approach to investigations of the near-boundary phenomena was proposed in [6], where the homogenization technique was applied independently to the near-boundary layer and interior part of a composite. Applications of the aforementioned asymptotic models lead to rather complicated computations, even for relatively simple problems, [4]. For more detailed review of papers on this subject the reader is referred to [8]. An alternative mathematical model for the analysis of the near-boundary and near-initial phenomena has been recently proposed in [7]. This model is based on the tolerance averaging technique of equations with periodic coefficients, [8]. The dynamic behaviour of a laminated solid was described in [7] by a refined homogenized equation of motion for the averaged displacements and by a certain boundary-layer equation for the displacement fluctuations. The above refined homogenized equations of motion involves the extra source term depending on displacement fluctuations but the boundary-layer equation is independent of the averaged displacements. This result has been obtained under a certain heuristic assumption neglecting terms which in most problems seem to be negligible small.

The aim of this contribution is to formulate mathematical models of the elastic periodically laminated solid which take into account some from results derived in [7] but reject the aforementioned heuristic assumption. The main result is that models proposed in this contribution describe not only the coupling between macro- and micro-response of the medium but also can be decomposed into successive asymptotic approximations. However, this decomposition holds only for a special class of laminated solids. The tolerance averaging equations which were derived in [8] are the starting point of the following considerations. In order to make this paper self consistent, in Section 2 we recall some from basic concepts of the tolerance averaging technique which can be found in monograph [8].

Denotations. Small bold-face letters stand for vectors and points in 3-space, capital bold-face letters denote second-order tensors and block letters are used for the third- and fourth-order tensors. The scalar and double-scalar products of these objects are denoted by the dot or the double dot between letters, respectively. The time derivatives are denoted by the over dot.

FUNDAMENTALS

Let $Ox_1x_2x_3$ stand for the orthogonal cartesian coordinate system in the physical space. Points of this space will be denoted by $\mathbf{x} = (x_1, x_2, x_3)$. Let $\Omega = (-L, L)\times\Pi$, $\Pi \subset R^2$ be the region in the physical space occupied by the laminated solid in its natural configuration. Interfaces between laminae are perpendicular to the Ox_1-axis. It is assumed that the solid is made of a large number of layers having equal thickness l, where $l << L$. Every layer is made of two homogeneous linear elastic laminae of thicknesses l' and l'', where $l' + l'' = l$. The mass density and the tensor of elastic moduli of laminae having thickness l' and l'' will be denoted by ρ', X' and ρ'', X'', respectively. It is also assumed that all laminae are perfectly bonded and every material plane x_1=const. is the elastic symmetry plane.

The starting point of our considerations are equations derived in [8] by the tolerance averaging technique. Following [8], for an arbitrary integrable function f defined on $[-L, L]$ we denote

$$\langle f\rangle(x_1) = \frac{1}{l}\int_{-l/2}^{l/2} f(x_1 + z)dz, \quad x_1 \in \left[-L + \frac{l}{2}, L - \frac{l}{2}\right]$$

where $l << L$ and f can also depend on x_2, x_3 and time t. Moreover, by $g(\cdot)$ we denote l-periodic, continuous function of argument x_1 such that $g(nl) = l\sqrt{3}$, $g(nl + l') = -l\sqrt{3}$, $n = 0,\pm1,\pm2,...$ and linear in every interval $[nl, nl + l']$ and $[nl + l', (n+1)l]$, $n = 0,\pm1,\pm2,...$. This function is referred to as the shape function of the two-component periodically laminated solid. Using this function we postulate that every displacement field $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ in problems under consideration can be restricted to the form given by

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) + g(x_1)\mathbf{v}(\mathbf{x}, t) \tag{1}$$

where **u** and **v** are slowly varying functions of argument x_1 in the sense explained in [9]. Functions **u** and **v** in (1) are the basic kinematical unknowns. Field **u** is the averaged displacement, $\mathbf{u} = \langle \mathbf{w}\rangle$, and **v** is referred to as the fluctuation amplitude.

Let us define $\nu' = l'/l$, $\nu'' = l''/l$, $\nu'\nu'' \neq 0$, $\mathbf{e} = (1,0,0)$ and $\nabla = (\partial_1, \partial_2, \partial_3)$, $\overline{\nabla} = (0, \partial_2, \partial_3)$ where $\partial_i = \partial / \partial x_i$, i=1,2,3. Setting aside all details related to the tolerance averaging technique it can be shown that the governing equations for **u** and **v** have the form

$$\begin{aligned} &\langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\langle \mathrm{X} \rangle : \nabla \mathbf{u} + [\mathrm{X}] \cdot \mathbf{v} \right) = \mathbf{0} \\ &l^2 \langle \rho \rangle \ddot{\mathbf{v}} - l^2 \overline{\nabla} \cdot \left(\langle \mathrm{X} \rangle : \overline{\nabla} \mathbf{v} \right) + \{\mathbf{C}\} \cdot \mathbf{v} + [\mathrm{X}]^T : \nabla \mathbf{u} = \mathbf{0} \end{aligned} \tag{2}$$

where the constant coefficients are given by

$$\langle \rho \rangle = \nu' \rho' + \nu'' \rho'', \quad \langle \mathrm{X} \rangle = \nu' \mathrm{X}' + \nu'' \mathrm{X}'', \quad [\mathrm{X}] = 2\sqrt{3} \left(\mathrm{X}'' - \mathrm{X}' \right) \cdot \mathbf{e},$$

$$[\mathrm{X}]^T = 2\sqrt{3} \mathbf{e} \cdot \left(\mathrm{X}'' - \mathrm{X}' \right), \quad \{\mathbf{C}\} = 12 \mathbf{e} \cdot \left(\frac{\mathrm{X}'}{\nu'} + \frac{\mathrm{X}''}{\nu''} \right) \cdot \mathbf{e}$$

Model equations (2) with formula (1) represent what is called the tolerance model of the periodically laminated two-component medium. Equations (2) have to be considered together with initial conditions for **u** and **v**, boundary condition for **u** on $\partial \Omega = \left((-L, L) \times \partial \Pi \right) \cup \left(\{-L\} \times \Pi \right) \cup \left(\{L\} \times \Pi \right)$ and boundary condition for **v** on $(-L, L) \times \partial \Pi$. The main feature of the above model is the coupling between macro- and micro-mechanics of a laminated medium. This coupling is represented by the presence of the microstructure length parameter l in the second from equations (2). Equations (1), (2) constitute the basis for the subsequent analysis. The detailed discussion of the above model equations can be found in [8].

ALTERNATIVE FORM OF MODEL EQUATIONS

The main aim of this contribution is to show that under certain conditions the model equations (2) can be decomposed into equations describing independently macro- and micro-response of the periodic laminate. To this end we shall derive an alternative form of the model equations (2). In this form, instead of fluctuation amplitude **v**, we shall deal with a new kinematical unknown **r** defined by

$$\mathbf{r} = \{\mathbf{C}\}^{-1} \cdot [\mathrm{X}]^T : \nabla \mathbf{u} + \mathbf{v} \tag{3}$$

where $\{\mathbf{C}\}^{-1}$ represents inverse to the non-singular linear transformation $\{\mathbf{C}\}$. For the homogenized model $\mathbf{r} \equiv \mathbf{0}$, [7]. That is why **r** is referred to as the intrinsic fluctuation amplitude, i.e., the amplitude independent of the averaged displacement field **u**. At the same time from (1) and (3) we obtain

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t) - g(x_1) \{\mathbf{C}\}^{-1} \cdot [\mathrm{X}]^T : \nabla \mathbf{u}(\mathbf{x}, t) + g(x_1) \mathbf{r}(\mathbf{x}, t)$$

where $g\mathbf{r}$ represents the intrinsic fluctuation of displacement. The part of response of periodic laminates described by the averaged displacement **u** is referred to as macro-response while that described by the intrinsic fluctuation amplitude **r** is called micro-response.

In order to formulate governing equations for functions **u** and **r** we shall use the notion of homogenized tensor of elastic moduli

$$\mathrm{X}^h \equiv \langle \mathrm{X} \rangle - [\mathrm{X}] \cdot \{\mathbf{C}\}^{-1} \cdot [\mathrm{X}]^T$$

We also introduce

$$A\mathbf{u} \equiv \langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\mathrm{X}^h : \nabla \mathbf{u} \right),$$

$$D\mathbf{r} \equiv l^2 \left[\langle \rho \rangle \ddot{\mathbf{r}} - \overline{\nabla} \cdot \left(\langle \mathrm{X} \rangle : \overline{\nabla} \mathbf{r} \right) \right] + \{\mathbf{C}\} \cdot \mathbf{r},$$

$$F\mathbf{u} \equiv l^2 \left[\langle \rho \rangle \{\mathbf{C}\}^{-1} \cdot [\mathrm{X}]^T : \nabla \ddot{\mathbf{u}} - \overline{\nabla} \cdot \left(\langle \mathrm{X} \rangle : \overline{\nabla} \cdot \left(\{\mathbf{C}\}^{-1} \cdot [\mathrm{X}]^T : \nabla \mathbf{u} \right) \right) \right]$$

Combining equations (2) with formula (3) we obtain the coupled system of the model equations for **u** and **r**

$$\begin{aligned} A\mathbf{u} &= [\mathrm{X}]^T : \nabla\mathbf{r} \\ D\mathbf{r} &= F\mathbf{u} \end{aligned} \tag{4}$$

which is an alternative to model equations (2). It has to be emphasized that equation (4) have a physical sense only if **u** and **r** are slowly varying functions of argument x_1. We recall that boundary conditions for **u** have to be prescribed on $(-L, L)\times\partial\Pi$ and on $\{-L, L\}\times\Pi$ while boundary conditions for **r** only on $(-L, L)\times\partial\Pi$.

The subsequent analysis of equations (4) will be based on the decomposition

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\Delta, \qquad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}_\Delta \tag{5}$$

where $\mathbf{u}_0$, $\mathbf{r}_0$ are assumed to satisfy equations

$$\begin{aligned} A\mathbf{u}_0 &= \mathbf{0}, \\ D\mathbf{r}_0 &= \mathbf{0} \end{aligned} \tag{6}$$

as well as boundary/initial conditions. These conditions in every problem under consideration have to be the same as those imposed on **u** and **r**, respectively. It follows that $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$ satisfy the corresponding homogeneous boundary/initial conditions and nonhomogeneous equations

$$\begin{aligned} A\mathbf{u}_\Delta &= [\mathrm{X}]^T : \nabla(\mathbf{r}_0 + \mathbf{r}_\Delta) \\ D\mathbf{r}_\Delta &= F(\mathbf{u}_0 + \mathbf{u}_\Delta) \end{aligned} \tag{7}$$

It has to be emphasized that the first from equations (6) represents the model obtained by the homogenization technique, [1,2], and the second from equations (6) describes phenomena related to the fluctuations of boundary and initial displacements, [7]. The analysis of equations (7) will be carried out in the subsequent section.

ASYMPTOTIC APPROXIMATIONS

Let us denote by $\|\cdot\|_n$ an arbitrary but fixed norm in the linear space of all *n*-th order tensors related to space E^3. Let us also define

$$\eta = \frac{\|[\mathrm{X}]\|_3}{\|\langle \mathrm{X}\rangle\|_4}$$

as a transversal inhomogeneity parameter of the laminates under consideration. These laminates are said to have a weak transversal inhomogeneity provided that η satisfies condition $0 < \eta <\!\!<\!\!< 1$. The above condition holds true for many laminated materials met in civil and mechanical engineering. The subsequent analysis will be restricted to laminates with a weak transversal inhomogeneity where η is treated as a certain small parameter.

Notice that the values of $[\mathrm{X}]^T : \nabla\mathbf{r}$ and $F\mathbf{u}$ are of an order $O(\mathbf{r}\eta)$, $O(\mathbf{u}\eta)$ and $A\mathbf{u}$, $D\mathbf{r}$ are of the same order as **u** and **r**, respectively. It can be shown that $\mathbf{u}_\Delta \in O(\eta)$, $\mathbf{r}_\Delta \in O(\eta)$. This result can be written in the asymptotic form

$$\mathbf{u}_\Delta = \mathbf{u}_1 + o(\eta), \quad \mathbf{r}_\Delta = \mathbf{r}_1 + o(\eta)$$

where $\mathbf{u}_1$, $\mathbf{r}_1$ are assumed to be linear functions of η. Applying the asymptotic limit passage $\eta \to 0$ to equations (7) we obtain the following system of equations for $\mathbf{u}_1$, $\mathbf{r}_1$:

$$A\mathbf{u}_1 = [\mathrm{X}]^T : \nabla\mathbf{r}_0$$
$$D\mathbf{r}_1 = F\mathbf{u}_0 \tag{8}$$

The above equations are assumed to hold together with homogeneous boundary and initial conditions. These conditions have the same form as pertinent homogeneous conditions for $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$, respectively. It can be seen that the right-hand sides of equations (8) are known provided that the boundary/initial value problem for equations (6) has been previously solved.

Summarising the obtained asymptotic results we state that model equations (4) for u and r can be decomposed to the form given by equations (6) and (8). In this case formula (5) yields

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + o(\eta), \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1 + o(\eta)$$

i.e. we deal with an asymptotic approximation of an order $o(\eta)$.

Equations (6) will be referred to as *the first order approximation model* for laminates with a weak transversal inhomogeneity. In the framework of this model the basic kinematical unknowns u, r are approximated by $\mathbf{u}_0$, $\mathbf{r}_0$, respectively. Equations (6) together with (8) will be referred to as *the second order approximation model*. In this model unknowns u, r are approximated by $\mathbf{u}_0 + \mathbf{u}_1$, $\mathbf{r}_0 + \mathbf{r}_1$, respectively. Formulation of higher-order approximation models is also possible and will be studied in a separate contribution.

ILLUSTRATIVE EXAMPLE

The general results of Sec. 3 and 4 will be now illustrated by a simple benchmark problem. We consider a thick unfinite laminated layer bounded by planes $x_2 = 0$, $x_2 = H$ normal to interfaces between laminae. It will be assumed that on boundary plane $x_2 = 0$ there are known constant values $\bar{\mathbf{u}} = (\bar{u},0,0)$ and $\bar{\mathbf{r}} = (\bar{r},0,0)$ of averaged displacements u and the fluctuation amplitudes r. On boundary plane $x_2 = H$ the values of fields u, r are assumed to be equal to zero. Bearing in mind the formula for total displacement field w it can be seen that from the physical viewpoint the above boundary conditions are rather artificial and have been introduced only to make subsequent calculations as simple as possible. Moreover, the problem will be treated as time independent. In this case, denoting $x \equiv x_2$, we obtain

$$\mathbf{u}(x) = (u(x), 0, 0)$$
$$\mathbf{r}(x) = (r(x), 0, 0), \qquad x \in [0, H]$$

Let G', G'' stand for the shear elastic modulae in both components of the laminate, such that $G'' > G'$. Correspondingly E', E'' stand for the longitudinal elastic modulae. Let us denote

$$\eta \equiv \frac{2\sqrt{3}(G'' - G')}{\nu'G' + \nu''G''},$$

$$\beta \equiv \frac{(\nu'G' + \nu''G'')(\nu'G'' + \nu''G')}{G'G''},$$

$$\alpha^2 \equiv \frac{12(\nu'G'' + \nu''G')}{\nu'\nu''(\nu'E' + \nu''E'')},$$

$$\gamma \equiv \frac{\nu'\nu''(\nu'G' + \nu''G'')}{12(\nu'G'' + \nu''G')}$$

where η is a transversal inhomogeneity parameter introduced in Sec. 4. It can be shown that, under the above notations, equations (4) take the form

$$\begin{aligned} u'' &= -\eta\beta r' \\ l^2 r'' - \alpha^2 r &= l^2 \eta\gamma u''' \end{aligned} \tag{9}$$

where unknowns $u(x),\ r(x),\ x \in [0, H]$ have to satisfy boundary conditions

$$u(0) = \bar{u},\quad u(H) = 0,\quad r(0) = \bar{r},\quad r(H) = 0 \tag{10}$$

We shall assume that $H >> l$ so that condition

$$1 + \exp\left(-\frac{H}{l}\right) \cong 1 \tag{11}$$

will take place in all subsequent manipulations.

The first order approximation model determined by equations (6) leads to the boundary value problem given by

$$u_0'' = 0, \qquad l^2 r_0'' - \alpha^2 r_0 = 0$$

$$u_0(0) = \bar{u},\quad u_0(H) = 0,\quad r_0(0) = \bar{r},\quad r_0(H) = 0$$

Hence

$$u_0 = \bar{u}\left(1 - \frac{x}{H}\right),\quad r_0 = \bar{r}\exp\left(-\alpha\frac{x}{l}\right)$$

The second order approximation model governed by equations (6) together with equations (8) yields

$$u_1'' = -\eta\beta r_0',\quad l^2 r_1'' - \alpha^2 r_1 = l^2 \eta\gamma u_0'''$$

$$u_1(0) = u_1(H) = 0,\quad r_1(0) = r_1(H) = 0$$

In this case we obtain

$$u_1 = \frac{\eta\beta}{\alpha} l\bar{r}\left[\frac{x}{H} - 1 + \exp\left(-\alpha\frac{x}{l}\right)\right],\quad r_1 = 0$$

Hence *u* and *r* are approximated by

$$\begin{aligned} u &\cong \bar{u}\left(1 - \frac{x}{H}\right) + \frac{\eta\beta}{\alpha} l\bar{r}\left[\frac{x}{H} - 1 + \exp\left(-\alpha\frac{x}{l}\right)\right] \\ r &\cong \bar{r}\exp\left(-\alpha\frac{x}{l}\right), \qquad x \in [0, H] \end{aligned} \tag{12}$$

with an approximation of an order $o(\eta)$.

Under the extra denotation $l_0 \equiv l\sqrt{1+\beta\gamma\eta^2}$ solution to equations (9) which satisfies boundary conditions (10) has the form

$$u = \bar{u}\left(1 - \frac{x}{H}\right) + \frac{\eta\beta}{\alpha} l_0 \bar{r}\left[\frac{x}{H} - 1 + \exp\left(-\alpha\frac{x}{l_0}\right)\right]$$
$$r = \bar{r}\exp\left(-\alpha\frac{x}{l_0}\right), \qquad x \in [0, H] \tag{13}$$

Apart from approximation (11), which for sufficiently small l / H has no essential meaning, formulae (13) can be treated as the exact solution to the boundary value problem under consideration. Comparing (12) and (13) we conclude that the second order approximation (12) describes the exact solution from the qualitative viewpoint. This situation does not hold for the first order approximation of averaged displacements. From the quantitative viewpoint the desired accuracy of the second order approximation is determined by the quotient l_0 / l. For a postulated value of l_0 / l (which should be close to 1) we can calculate η and hence to formulate a necessary condition imposed on laminates with a weak transversal inhomogeneity described by the second order approximation model.

CONCLUSIONS

The main results and new information about modelling of periodically laminated solids presented in this contribution can be stated as follows

1. Following [7], an alternative form (4) of the tolerance model equations (2) derived in [8] was introduced. The basic kinematical unknowns in equations (4) are: averaged displacement **u** and intrinsic fluctuation amplitude **r**. The intrinsic fluctuation describes a micro mechanical response of a laminate. In the framework of the well known homogenized model of a laminated medium this response is neglected.
2. The concept of laminates with a weak transversal inhomogeneity was introduced. This concept is based on the proposed definition of the transversal inhomogeneity parameter η. This kind of inhomogeneity takes place for laminae reinforced by long high-strength fibres. In this case components of elastic moduli tensor X which are related to the Ox_2x_3-plane are strongly different in adjacent laminae; the remaining components attain only small jumps across the lamina interfaces.
3. It was shown that for laminates with a weak transversal inhomogeneity the model equations (4) for **u** and **r** can be decomposed into successively independent equations (6) and (8) for new unknowns $\mathbf{u}_0$, $\mathbf{r}_0$, $\mathbf{u}_1$, $\mathbf{r}_1$. Equations (6) for $\mathbf{u}_0$, $\mathbf{r}_0$ represent the first order approximation model. In the framework of this model unknowns **u, r** are approximated by $\mathbf{u}_0$, $\mathbf{r}_0$, respectively, with an asymptotic approximation of an order $O(\eta)$. Equations (6) for $\mathbf{u}_0$, $\mathbf{r}_0$ together with equations (8) for $\mathbf{u}_1$, $\mathbf{r}_1$ constitute the second order approximation model. Using this model we approximate **u** and **r** by $\mathbf{u}_0 + \mathbf{u}_1$ and $\mathbf{r}_0 + \mathbf{r}_1$, respectively, with an asymptotic approximation of an order $o(\eta)$.
4. The final conclusion is that for laminates with a weak transversal inhomogeneity the model equations (4) can be decoupled into equations for $\mathbf{u}_0$, $\mathbf{r}_0$, $\mathbf{u}_1$, $\mathbf{r}_1$ which can be solved successively. Let us observe that in equations (4) micro mechanical response **r** is coupled with averaged displacement **u**.

The general results have been illustrated by a simple benchmark problem where solution obtained in the framework of the second order approximation model was compared with the exact solution. This comparison makes it possible to evaluate the effect of the transversal inhomogeneity parameter η on the accuracy of approximate solutions. It was shown that for η small when compared to 1 the second order approximation model yields reliable results both from the qualitative and quantitative viewpoint. On the other hand, the first order approximation model is unable to describe properly the problem under consideration.

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