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ON THE DYNAMIC BEHAVIOUR OF UNIPERIODIC PLATES MADE OF ORTHOTROPIC ELEMENTS

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ABSTRACT

The aim of this contribution is to propose a new non-asymptotic 2D-model of non-homogeneous Reissner type elastic plates with one-directional periodic (uniperiodic) structure. This model was obtained by tolerance averaging technique (TAA) describing effect of repetitive cell size l (and at the same time the period-length of in-homogeneity) on the overall plate behaviour. The new feature of the proposed model is the possibility to apply the analysis of plates having thickness of an order of the period-length. So far, the non-asymptotic 2Dmodel of Reissner-type uniperiodic plates was formulated under assumption that the plate thickness is very small compared to the period-length.

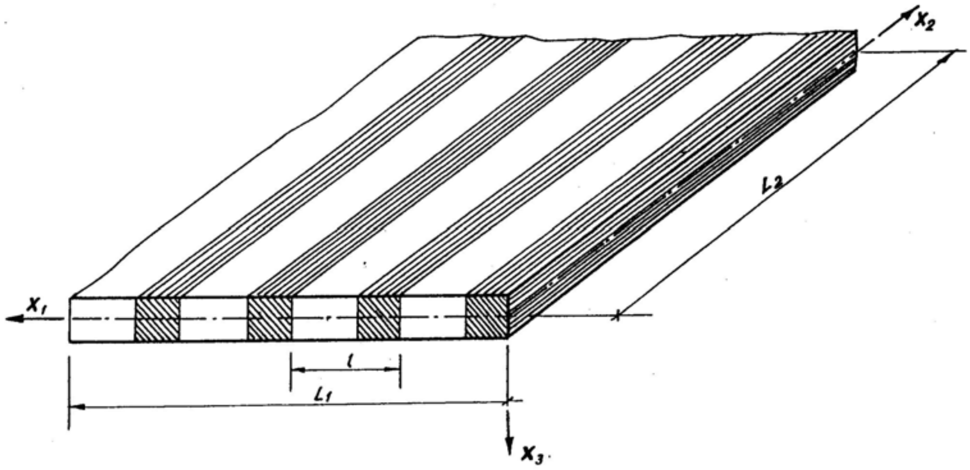
Keywords: uniperiodic composite plates, Reissner type elastic plates, tolerance averaging technique, modeling, dynamic.

1. INTRODUCTION

Plates with uniperiodic structure could also be used in civil engineering. There are first of all composite plates. However, plates made of traditional materials represent such structure. The concrete plate reinforced by system of rolled steel section (*I*-bar) can be example of this case. In this paper, the example of glued timber plate (that is composed of the elements cut along and across the fibres) was considered.

The object of consideration is the description of dynamic behaviour of a medium thickness uniperiodic, elastic composite plates, i.e. plates with a periodic non-homogeneous structure in one direction. The above plates are composed of a large number of repeated elements having identical form, dimensions and material properties. The geometry of the uniperiodic plate, apart from the global midplane length dimensions L_1, L_2 and constant thickness d, is characterized by the length l which determines the period of structure in-homogeneity. A fragment of the aforementioned plate is shown in Fig.1.

Fig. 1. An example of medium thickness uniperiodic plate



The dynamic behaviour of uniperiodic composite plates is described by partial differential equations (PDEs) with functional coefficients which are periodic, rapidly oscillating, and discontinuous with respect to one Cartesian coordinate (say, the x_1-coordinate) along which the plate structure is periodic. The direct application of these equations to the analysis of special problems in most cases is not possible. It is known that even a direct numerical solution of these equations constitutes an ill-conditioned and complicated problem, cf. [4]. The fundamental modelling problem in how to obtain an effective 2D-model represented by PDEs with constant coefficients. This problem has been solved by using the homogenization theory [5]. [7]; homogenized models of medium thickness periodic plates were studied in a series of papers; s.e.g. [9], [10], [11]. On the other hand, the use of asymptotic homogenization method results in neglecting the length – scale effect, i.e. the effect of the period l on the macrodynamic plate behaviour. In many dynamic problems, however, this effect cannot be neglected, therefore a new non-asymptotic approach to the modelling of plates with periodic structure has been presented in [14]. This approach is a certain generalization of the tolerance averaging technique [13], which makes it possible to investigate the length – scale effect. So far, the tolerance averaging technique was applied in the modelling of medium thickness uniperiodic plates under assumption that the period l is very large as compared to the maximum plate thickness [1], [2], [3]. The main aim of this paper is to formulate a new non-asymptotic model of Reissner – type uniperiodic plates with period of an order of the plate thickness. Considering the above assumptions, only the thin plates have been modelled so far, [12], [14]. The non-asymptotic effective 2D – model of Reissner – type plates with bi-directional periodic structure having thickness of an order of the plate periods length has been discussed in [15].

The proposed model will be obtained by the tolerance averaging technique, applied directly to the 3D – equations of linear elastodynamics. Using the Hencky – Bolle kinematics assumption, we shall derive a non–asymptotic 2D–model of medium thickness uniperiodic plates. Contrary to the homogenized 2D–model, it makes possible to determine higher–order vibration frequencies caused by the plate inhomogeneity. It should be noticed, that in general, plates with uniperiodic structure cannot be treated as a special case of plates with bidirectional periodicity [15].

The general model equations obtained in this paper will be transformed into a form which would enable the investigation of dynamic problems for uniperiodic composite plates made of orthotropic elements. The considerations are illustrated by the analysis of vibrations of rectangular plate.

Throughout the paper, subscripts $\alpha, \beta,...$ $(i, j,...)$ run over 1,2 (1,2,3), whereas superscripts A,B,... take the values 1,2,...N. The summation convention holds for all aforementioned indices.

2. BASIC ASSUMPTIONS AND DENOTATIONS

Let Ox_1x_2 be the orthogonal Cartesian coordinate system in the physical space E^3, and Ω are a region occupied by the solid under consideration in its reference state. Let $\Delta(\mathbf{x}) = \Delta + \mathbf{x}$ be a periodic cell of the central of point $\mathbf{x} = \{x_i\} \in E^3$. By l_i we denote the period of the solid inhomogeneity in direction of the x_i – axis. It will be assumed that l_i are sufficiently small when compared to the characteristic length dimension of Ω

measured along the x_i – axis. It is possible to consider three special cases of periodic inhomogeneity, c.f. [14]. In this paper, considerations will be restricted to the bending of plates with uniperiodic structure. Therefore, for a solid periodic in direction of the x_1 – axis $\Delta = \Delta_1 = (-l/2, l/2)$ is the periodicity interval and $l = l_1$ is the period of in-homogeneity. The averaged value of an arbitrary integrable function $f(\mathbf{x})$ (defined on Ω) inside the periodicity interval is denoted as

$$\left\langle f(\mathbf{x})\right\rangle=\frac{1}{l}\int_{x_1-l/2}^{x_1+l/2} f(y_1,x_2,x_3)dy_1 \quad \mathbf{x}\in \Omega_o, \;\; \Omega_o=\left\{\mathbf{x}\in E^3, \Delta(\mathbf{x})\subset\Omega\right\} \tag{2.1}$$

The tolerance averaging technique is based on the definition of the averaging operator (2.1) and concept of a slowly varying function of an argument $\mathbf{x}$. For solid with a uniperiodic structure, it is a slowly-varying function of an argument x_1, we shall mean a sufficiently regular function F(x), which for an arbitrary integrable function $f(\mathbf{x})$ satisfying the following tolerance averaging approximation (TAA)

$$\left\langle fF\right\rangle(\mathbf{x}) \cong \left\langle f\right\rangle(\mathbf{x})F(\mathbf{x}) \tag{2.2}$$

where $\cong$ is a certain tolerance relation [13]. If condition (2.2) holds for all continuous derivatives of F (if they exist) then we shall write $F(\cdot)\in SV_{\Delta}(T)$. By T we denote the set of all tolerance relations regarded in the modelling procedure.

Let $u_i(\mathbf{x},t), \mathbf{x}\in\Omega$ be a displacement field at time t from the reference configuration of the uniperiodic elastic solid. For solid of this structure, we assume the mass density scalar field ρ and the components A_{ijkl} of the elastic modulae tensor are l-periodic function of x_1-coordinate and are independent of x_2-, and x_3-coordinate. We assume also that the solid is subjected to initial stress σ^{o}_{ij} , b_i is a constant body force field.

From the principle of stationary action for the functional depending on the displacement field components we obtain the following linearized equations of motion for an uniperiodic prestressed solid.

$$\left(A_{ijkl}u_{k,l} \right)_{,j} + \sigma^{o}_{kj}u_{i,jk} - \rho\ddot{u}_i + \rho b_i = 0 \tag{2.3}$$

Equations (2.3) has functional coefficients A_{ijkl} and ρ which are highly oscillating (frequently non-continuous) with respect to the argument x_1. In most cases the prestressing field tensor σ^{o}_{ij} is also periodic and non-continuous.

The modelling problem we are going to solve is to derive from (2.3) a system of equations with constant coefficients (differently – independent of x_1).

3. MODELLING APPROACH

The starting point of consideration is Eqs. (2.3). The modelling procedure is based on the tolerance averaging of the above equations. We are to formulate an approximate model of the uniperiodic solid, which will be represented by equations with constant coefficients. The proposed modelling technique is based on two assumptions. To formulate these assumptions we introduce the following decomposition of displacements:

$$u_i(\mathbf{x},t) = u^{o}_i(\mathbf{x},t) + r_i(\mathbf{x},t) \tag{3.1}$$

where $u^{o}_i(\mathbf{x},t)=\left\langle u_i\right\rangle(\mathbf{x},t)=\left\langle \rho\right\rangle^{-1}\left\langle \rho u_i\right\rangle(\mathbf{x},t)$ is an averaged part of displacement and $r_i(\cdot,t)$ is a part of residual displacement field.

The first modelling assumption states that averaged displacement field under consideration are a slowly varying functions for every time *t*: $u_i^o(\cdot,x_2,x_3,t)\in SV_\Delta(T)$. Under this assumption, from (3.1) and (2.2) it follows that $\langle \rho r_i \rangle(\mathbf{x},t)=0$ and that is why r_i will be referred to as the fluctuating part of displacements.

The second modelling assumption states that the fluctuation of displacements field, represented by r_i and caused by the inhomogeneous periodic structure, conforms to this structure. It means that in every periodicity interval fluctuations r_i can be approximated by periodic functions in the form of finite sums

$$r_i(\mathbf{x},t) \cong h_i^A(x_i)V^A(\mathbf{x},t) \tag{3.2}$$

where $V^A(\cdot,x_2,x_3,t)$ for every time t are slowly varying functions $V^A(\cdot,x_2,x_3,t)\in SV_\Delta(T)$ and $h_i^A(x_1)$ are certain linear independent periodic functions satisfying the conditions $\langle \rho h_i^A \rangle=0,\ h_i^A(x_1)\in O(l),\ l h_{i,1}^A\in O(l)$. The approximation ⊔ depends on the number of terms on the right-hand sides of (3.2).

Scalar functions $V^A(\cdot,x_2,x_3,t)$ constitute new kinematical variables called fluctuation amplitudes and are the basic unknowns. Functions $h_i^A(x_1)$ are assumed to known *a priori* and are referred to as mode-shape function. In general, $h_i^A(\cdot)$ represent free periodic vibrations of 3D-periodic cell and can be treated as eigenvector related to certain eigenvalue problem. An alternative specification of the mode-shape function based on the mass discretization of the periodic cell is also possible.

In order to derive the governing equation for unknown fields u_i^o, V^A we shall introduce into the action functional displacement field u_i in the form

$$u_i(\mathbf{x},t)=u_i^o(\mathbf{x},t)+h_i^A(x_1)V^A(\mathbf{x},t) \tag{3.3}$$

Applying the principle of stationary action and averaging the obtained result, taking into account TAA (2.2), restrict consideration to problem in which $A_{ijkl}(\cdot)$ and $\rho(\cdot)$ are even and $h_i^A(\cdot)$ are odd functions, after some manipulation we obtain the following system of equations for u_i^o and V^A :

$$\begin{aligned}
&\langle \rho \rangle \ddot{u}_i^o-\left(\langle A_{ijkl}\rangle u_{k,l}^o+\langle A_{ikl1}h_{k,1}^A\rangle V^A\right)_{,j}-\langle \sigma_{kl}^o\rangle u_{i,kl}^o-\langle \rho \rangle b_i=0\\
&\langle \rho h_i^A h_i^B\rangle \ddot{V}^B-\langle A_{i2k2}h_i^A h_k^B\rangle V_{,22}^B-\langle A_{i3k3}h_i^A h_k^B\rangle V_{,33}^B+\\
&+\langle A_{i1k1}h_{i,1}^A h_{k,1}^B\rangle \mathrm{V}^B+\langle A_{ijk1}h_{k,1}^A\rangle u_{i,j}^o+\langle \sigma_{11}^o h_{k,1}^A h_{k,1}^B\rangle V^B+\\
&-\langle \sigma_{22}^o h_k^A h_k^B\rangle V_{,22}^B-\langle \sigma_{33}^o h_k^A h_k^B\rangle V_{,33}^B=0
\end{aligned} \tag{3.4}$$

Formulae (3.4) represent the system 3+*N* equations for 3+*N* unknown function $u_i^o(\mathbf{x},t), V^A(\mathbf{x},t),\ \mathbf{x}\in \Omega_o$. The equations have constant coefficients and represent a certain macroscopic model of a prestressed uniperiodic solid. It has to be emphasized that solutions u_i^o, V^A have physical sense only if they are represented by slowly varying function. Let us observe that $\langle \rho h_k^A h_k^B \rangle,\ \langle A_{i3k3}h_i^A h_k^B\rangle$ depend on the period *l* and describe the length scale effect on the overall behaviour of the solid.

For homogeneous solid, in the absence of initial stress, equation $(3.4)_1$ lead to the well known form and $(3.4)_2$ yields $V^A=0$ provided that initial as well as boundary condition for V^A are homogeneous. It becomes remarkable that for uniperiodic solid in which $\Omega=(0,L_1)\times\Phi,\ \ \Phi\subset R^2$, the boundary conditions for V^A can be formulated only on boundaries $(0,L_1)\times\partial\Phi$.

4. APPLICATION TO MEDIUM THICKNESS PLATES

Setting now $\boldsymbol{x} = (x_1,x_2)$, $z = x_3$, we assume that equations (3.4) hold in a region Ω, occupied by a Reissner-type undeformed plate with constant thickness d, defined by $\Omega = \{(\boldsymbol{x}, z) : |z| < d/2,\ \boldsymbol{x} \in \Pi\}$ where rectangular $\Pi = (0, L_1) \times (0, L_2)$ is the plate midplane.

The plate under consideration has a periodic non-homogeneous structure, with the period $l = l_1$, only in the direction of the x_1 – axis, i.e. uniperiodic structure. Therefore $(-l/2 + x_1,\ x_1 + l/2)$, for every $x_1 \in (l/2,\ L_1 - l/2)$ is the periodicity interval which has its centre at an arbitrary point with the x_1 – axis. In this case, the periodicity cell is defined by $(x_1 - l/2\,, x_1 + l/2) \times (-d/2\,, d/2)$ The dimension l is of an order the plate thickness d and sufficiently small compared to L_α, $l << L_\alpha$. The plate material is assumed to be elastic and the components A_{ijkl} of the elastic modulae tensor as well the mass density ρ independent on x_2, z and l – periodic function with respect to x_1 coordinate. Subsequently we define

$$C_{\alpha\beta\gamma\delta} = A_{\alpha\beta\gamma\delta} - A_{\alpha\beta 33} A_{33\gamma\delta} (A_{3333})^{-1},\ B_{\alpha\beta} = K\, A_{\alpha 3\beta 3}$$

where K is the shear coefficient of the medium-thickness plate theory,

Instead the operator (2.1) we introduce the following two kinds of averaging of an arbitrary integrable function $f(\boldsymbol{x},z,t)$ inside periodicity cell:

$$\begin{aligned} \langle f \rangle(\boldsymbol{x},z,t) &= \frac{1}{l} \int_{x_1 - l/2}^{x_1 + l/2} f(y_1, x_2, z, t)\, dy_1, \\ \langle f \rangle(\boldsymbol{x},t) &= \frac{1}{d} \int_{-d/2}^{d/2} \langle f \rangle(\boldsymbol{x},z,t)\, dz \end{aligned} \tag{4.1}$$

for l – periodic function, the averaged $(4.1)_1$ is independent of x_1.

We introduce the Hencky-Bolle kinematical assumption in the known form

$$\begin{aligned} u_\alpha(\boldsymbol{x},z,t) &= z\vartheta_\alpha(\boldsymbol{x},t) \\ u_3(\boldsymbol{x},z,t) &= w(\boldsymbol{x},t) \end{aligned} \tag{4.2}$$

where $w(\cdot,t)$ is displacements of point of the mid-plane Π and $\vartheta_\alpha(\cdot,t)$ are independent rotations.

According to the modelling assumption, outlined in the section 3, exist decompositions of w and ϑ_α into slowly varying parts w^o, ϑ^o_α and to residual displacement parts approximated by finite sums $h^A_i(x_1) V^A(\boldsymbol{x},z,t)$. Assuming that $h^A_3(x_1) = 0$ and $V^A(\boldsymbol{x},z,t) = z\psi^A(\boldsymbol{x},t)$ we obtain the components of displacement field in the form

$$\begin{aligned} u_\alpha(\boldsymbol{x},z,t) &= z\vartheta^o_\alpha(\boldsymbol{x},t) + z h^A_\alpha(x_1)\psi^A(\boldsymbol{x},t) \\ u_3(\boldsymbol{x},z,t) &= w^o(\boldsymbol{x},t) \end{aligned} \tag{4.3}$$

where the averaged midplane deflection $w^o(\boldsymbol{x},t)$, averaged rotations $\vartheta^o_\alpha(\boldsymbol{x},t)$ and fluctuation amplitudes $\psi^A(\boldsymbol{x},t)$ are slowly varying function and constitute a system of new basic unknowns.

Substituting the decomposition (4.3) into the action functional, neglecting the body forces and denoted $j = d^2/12$, after some manipulations we arrive at the system of equations

$$
\begin{aligned}
&j\langle \rho\rangle \ddot{\vartheta}^{o}_{\alpha} - j\langle C_{\alpha\beta\gamma\delta}\rangle \vartheta^{o}_{\gamma,\beta\delta} + \langle B_{\alpha\beta}\rangle(\vartheta^{o}_{\beta}+w^{o}_{,\beta}) + \\
&-j\langle C_{\alpha\beta\gamma 1}h^{A}_{\gamma,1}\rangle \psi^{A}_{,\beta} - j\langle \sigma^{o}_{\gamma\beta}\rangle \vartheta^{o}_{\alpha,\gamma\delta} + \langle z\sigma^{o}_{\gamma 3}\rangle \vartheta^{o}_{\alpha,\gamma} = 0 \\
&\langle \rho\rangle \ddot{w}^{o} - \langle B_{\alpha\beta}\rangle(\vartheta^{o}_{\alpha}+w^{o}_{,\alpha})_{,\beta} - \langle \sigma^{o}_{\gamma\alpha}\rangle w^{o}_{,\gamma\beta} = 0 \\
&j\langle \rho h^{A}_{\alpha}h^{B}_{\alpha}\rangle \ddot{\psi}^{B} + \left(j\langle C_{\alpha 1\gamma 1}h^{A}_{\alpha,1}h^{B}_{\gamma,1}\rangle + \langle B_{\alpha\beta}h^{A}_{\alpha}h^{B}_{\beta}\rangle + \langle C_{\alpha 2\gamma 2}h^{A}_{\alpha}h^{B}_{\gamma}\rangle\right)\psi^{B} + \\
&-j\langle C_{\alpha 2\gamma 2}h^{A}_{\alpha}h^{B}_{\gamma}\rangle \psi^{B}_{,22} + j\langle C_{\alpha\beta\gamma 1}h^{A}_{\gamma,1}\rangle \vartheta^{o}_{\alpha,\beta} + j\langle \sigma_{11}h^{A}_{\gamma,1}h^{B}_{\gamma,1}\rangle \psi^{B} - j\langle \sigma_{22}h^{A}_{\gamma}h^{B}_{\gamma}\rangle \psi^{B}_{,22} = 0
\end{aligned}
\tag{4.4}
$$

The system 3+N equations with constant coefficients (4.4), represent the macroscopic 2D – model of the uniperiodic, medium thickness composite plate. Those equations have to be considered together with two initial conditions for w^{o}, ϑ^{o}_{α} and ψ^{A}, which are obtained from the conditions for u_{α}, by means of formulae (4.3). Moreover, the averaged plate deflection w^{o} and averaged rotations ϑ^{o}_{α} have to satisfy boundary condition in the form similar to that used in the Reissner plate theory. On the other hand, boundary conditions for ψ^{A} have to be known only for edges $x_2 = 0$, $x_2 = L_2$. It follows that for edges $x_1 = 0$, $x_1 = L_1$, the boundary conditions can be satisfied only by the averaged part of the rotations ϑ^{o}_{α}.

Let use observe that the coefficients $\langle \rho h^{A}_{\alpha}h^{B}_{\alpha}\rangle$, $\langle C_{\alpha 2\gamma 2}h^{A}_{\alpha}h^{B}_{\gamma}\rangle$, $\langle B_{\alpha\beta}h^{A}_{\alpha}h^{B}_{\beta}\rangle$, $\langle \sigma_{22}h^{A}_{\gamma}h^{B}_{\gamma}\rangle$ depend on the period l (strictly l^2), thus, the equations (4.4) describe the length scale effect on the overall behaviour of the plate. Neglecting terms involving the period l lead to the system of linear algebraic equation for ψ^{A}. In this case, ψ^{A} do not enter the boundary conditions and play role of certain internal variables. In general, those internal variables can be eliminated from (4.4) and we obtain the system of three equations for w^{o} and ϑ^{o}_{α} as the basic unknowns. This system representing certain approximation of the homogenized 2D–model of the uniperiodic plate under consideration. Notice that, for a homogeneous plate, after setting $l \to 0$ and neglecting the initial stress, we obtain $\psi^{A} = 0$ and (4.4) coincide with the known Hencky-Bolle plate equation.

As is was mentioned, equations (4.4) represent the non-asymptotic 2D – model of the Reissner – type plate with uniperiodic non-homogeneous structure. This model was obtained by the tolerance averaging technique applied directly to the 3D – equations of elastodynamics, in contrast to the approaches proposed in [1], [2], [6], where 2D – equation of plates with oscillating coefficients together with assumption that the period l is very large to the plate thickness d, have been used as a starting point of modelling. The obtained equations (4.4) can be applied to problems in which the period l is of the same order as the plate thickness. At the same time it constitutes the basis for the subsequent analysis.

5. AN UNIPERIODIC PLATE MADE OF ORTHOTROPIC ELEMENTS

5.1 Model equation

Now let us assume that the plate consists of an orthotropic, homogeneous and elastic materials (periodically spaced along the x_1 – axis), for which the principal axis of orthotropy coincides with the Cartesian axis (**x**, z). An example of the uniperiodic plate made of two materials is shown in Fig.1. Taking into account the orthotropy of the plate materials we denote the non-zero components of the elastic moduli tensor:

$C_{11} = C_{1111}$ $\quad$ $C_{22} = C_{2222}$
$C_{12} = C_{1122} = C_{2211}$ $\quad$ $C = C_{1212} = C_{1221} = C_{2121} = C_{2112}$
$D_1 = B_{11}$ $\quad$ $D_2 = B_{22}$.

Let us take exclusively only one vector shape functions $h_1(x_1)=h^{1}_{1}(x_1), h_2(x_1)=h^{1}_{2}(x_1)$ as the first approximation of the plate fluctuation caused by the uniperiodic plate structure. Under this condition, the components of displacement field have a form

$$
\begin{aligned}
&u_{\alpha}(\mathbf{x},z,t) = z\vartheta^{o}_{\alpha}(\mathbf{x},t) + zh_{\alpha}(x_1)\psi(\mathbf{x},z,t) \\
&u_{3}(\mathbf{x},z,t) = w^{o}(\mathbf{x},t)
\end{aligned}
\tag{5.1}
$$

On the above assumptions, we obtain from (4.4) the following system of equations for the unknowns w^{o}, ϑ_{α}^{o} and $\psi = \psi^{1}$

$$\begin{aligned}
& j\langle \rho \rangle \ddot{\vartheta}_1^{o} - \left[j\langle C_{11} \rangle \vartheta_{1,11}^{o} + j\left(\langle C_{12} \rangle + \langle C \rangle \right) \vartheta_{2,12}^{o} + j\langle C \rangle \vartheta_{1,22}^{o} \right] + \langle D_1 \rangle \left(\vartheta_1^{o} + w_{,1}^{o} \right) + \\
& -\left(j\langle C_{11} h_{1,1} \rangle \psi_{,1} + j\langle C h_{2,1} \rangle \psi_{,2} \right) - j\langle \sigma_{\alpha\beta}^{o} \rangle \vartheta_{1,\alpha\beta}^{o} + \langle z\sigma_{\alpha 3}^{o} \rangle \vartheta_{1,\alpha}^{o} = 0, \\
& j\langle \rho \rangle \ddot{\vartheta}_2^{o} - \left[j\langle C_{22} \rangle \vartheta_{2,22}^{o} + j\left(\langle C_{12} \rangle + \langle C \rangle \right) \vartheta_{1,12}^{o} + j\langle C \rangle \vartheta_{2,11}^{o} \right] + \langle D_2 \rangle \left(\vartheta_2^{o} + w_{,2}^{o} \right) + \\
& -\left(j\langle C_{22} h_{1,1} \rangle \psi_{,2} + j\langle C h_{2,1} \rangle \psi_{,1} \right) - j\langle \sigma_{\alpha\beta}^{o} \rangle \vartheta_{2,\alpha\beta}^{o} + \langle z\sigma_{\alpha 3}^{o} \rangle \vartheta_{2,\alpha}^{o} = 0, \\
& \langle \rho \rangle \ddot{w}^{o} - \langle D_1 \rangle \left(\vartheta_1^{o} + w_{,1}^{o} \right)_{,1} - \langle D_2 \rangle \left(\vartheta_2^{o} + w_{,2}^{o} \right)_{,2} - \langle \sigma_{\alpha\beta}^{o} \rangle w_{,\alpha\beta}^{o} = 0, \\
& j\left\langle \rho\left(h_1^2 + h_2^2 \right) \right\rangle \ddot{\psi} + \left(j\langle C_{11} h_{1,1}^2 \rangle + j\langle C h_{2,1}^2 \rangle + \langle D_1 h_1^2 \rangle + \langle D_2 h_2^2 \rangle \right) \psi + \\
& -\left(j\langle C_{22} \mathrm{h}_2^2 \rangle + j\langle C h_1^2 \rangle \right) \psi_{,22} + j\langle C_{11} h_{1,1} \rangle \vartheta_{1,1}^{o} + j\langle C_{12} h_{1,1} \rangle \vartheta_{2,2}^{o} + \\
& + j\langle C h_{2,1}^{o} \rangle \left(\vartheta_{1,2}^{o} + \vartheta_{2,1}^{o} \right) + j\left\langle \sigma_{11}^{o} \left(h_{1,1}^2 + h_{2,1}^2 \right) \right\rangle \psi - j\left\langle \sigma_{22}^{o} \left(h_1^2 + h_2^2 \right) \right\rangle \psi_{,22} = 0.
\end{aligned} \tag{5.2}$$

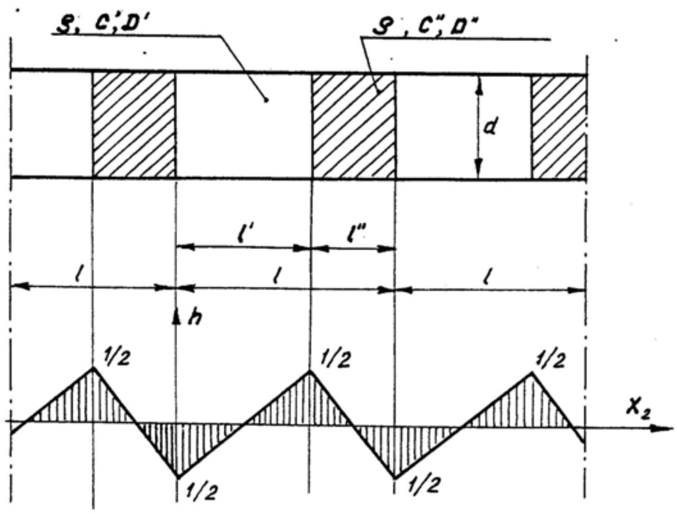
Equations (5.2), together with formulae (5.1), constitute the proposed non-asymptotic 2D– model of the medium thickness uniperiodic composite plates made of orthotropic components. At the same time they represent a certain first approximation of governed Eqs. (4.4). This model enables analysing of many dynamic and stability problems.

5.2 Analysis of free vibrations

As an example of application of the Eqs. (5.2), free vibrations of a rectangular, simply supported on its edges plate with uniperiodic structure will be studied. It will be assumed that the plate is made only of single orthotropic and homogeneous material. The repetitive segment (periodicity cell) of this plate consist of two parts in which the principal axis of orthotropy in the mid-plane, are turned by 90^{o}. The plate with such a structure, so called uniperiodic with respect to the elastical properties. Under these conditions in equations (5.2) the coefficients are $\langle C_{12} h_{1,1} \rangle = \langle C h_{2,1} \rangle = 0$.

To simplify calculations let's assume that the effect of initial stress on the free vibrations can be neglected and the mode shape function $h_1 = h_2 = h$ where h is a periodic saw-like function, the diagram of which is shown in Fig.2.

Fig.2. Periodicty cell and diagrams of function $h(\cdot)$



Taking into account the boundary condition we look for the solution to equations (5.2) in the form

$$
\begin{aligned}
\vartheta_1^{o} &= e^{i\omega t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\vartheta_{1mn} cos\alpha_m x_1 sin\beta_n x_2,\\
\vartheta_2^{o} &= e^{i\omega t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\vartheta_{2mn} sin\alpha_m x_1 cos\beta_n x_2,\\
w^{o} &= e^{i\omega t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} sin\alpha_m x_1 sin\beta_n x_2,\\
\psi &= e^{i\omega t}\sum_{n=1}^{\infty}\overline{\psi}_n(x_1) sin\beta_n x_2,
\end{aligned}
\tag{5.3}
$$

where $\alpha_m = \frac{m\pi}{L_1}$, $\beta_n = \frac{n\pi}{L_2}$, $m,n=1,2,...$ and $\vartheta_{1mn}, \vartheta_{2mn}, w_{mn}$ are constant amplitudes, ω is a vibration frequency. For ψ boundary conditions are known only on edges $x_2 = 0$, $x_2 = L_2$.

Denoting

$$
B = j\left(\left\langle C_{11}h_{,1}^2\right\rangle + \left\langle \mathrm{Ch}_{,1}^2\right\rangle\right) + \left\langle D_1 h^2\right\rangle + \left\langle D_2 h^2\right\rangle + \beta_n^2 j\left(\left\langle C_{22}h^2\right\rangle + \left\langle Ch^2\right\rangle\right)
$$

we obtain from $(5.2)_4$

$$
\overline{\psi}_n(x_1) = \frac{j\left\langle C_{11}h_{,1}\right\rangle}{B - 2\left\langle \rho h^2\right\rangle\omega^2}\sum_{m=1}^{\infty}\alpha_m\vartheta_{1mn} sin\alpha_m x_1
\tag{5.4}
$$

thus, the unknown $\overline{\psi}_n(x_1)$ can be eliminated from the model equation.

Substituting (5.3) into (5.2), taking into account the aforementioned assumption and (5.4), after denotation

$$
H_\omega = \left\langle C_{11}\right\rangle - j\frac{\left\langle C_{11}h_{,1}\right\rangle^2}{B - 2\left\langle \rho h^2\right\rangle\omega^2} \quad ,
$$

we obtain the following system of three linear algebraic equation for constant amplitudes

ϑ_{1mn} , ϑ_{2mn} , w_{mn} :

$$
\begin{bmatrix}
\alpha_m^2 jH_\omega + \beta_n^2 j\left\langle C\right\rangle + \left\langle D_1\right\rangle - j\left\langle \rho\right\rangle\omega^2 & \alpha_m\beta_n j\left(\left\langle C_{12}\right\rangle + \left\langle C\right\rangle\right) & \alpha_m\left\langle D_1\right\rangle \\
\alpha_m\beta_n j\left(\left\langle C_{12}\right\rangle + \left\langle C\right\rangle\right) & \beta_n^2 j\left\langle C_{22}\right\rangle + \alpha_m^2\left\langle C\right\rangle + \left\langle D_2\right\rangle - j\left\langle \rho\right\rangle\omega^2 & \beta_n\left\langle D_2\right\rangle \\
\alpha_m\left\langle D_1\right\rangle & \beta_n\left\langle D_2\right\rangle & \alpha_m^2\left\langle D_1\right\rangle + \beta_n^2\left\langle D_2\right\rangle - \left\langle \rho\right\rangle\omega^2
\end{bmatrix}
\begin{bmatrix}\vartheta_{1mn}\\ \vartheta_{2mn}\\ w_{mn}\end{bmatrix}
=
\begin{bmatrix}0\\0\\0\end{bmatrix}
\tag{5.5}
$$

and additional condition $B \neq 2\left\langle \rho h^2\right\rangle\omega^2$.

One should pay attention, that for plate described by Eqs. (5.5), coefficients $\left\langle C\right\rangle$ and $\left\langle C_{12}\right\rangle$ not depend to configuration of components on the periodicity cell. Moreover, the averaged include function h^2, for example

$\langle D_1 h^2 \rangle$, one can present in the form $\langle D_1 h^2 \rangle = l^2/12 \langle D_1 \rangle$.This coefficient depends *explicite* on the period l and describes the length-scale effect.

Bearing in mind, that the period l is of an order of the plate thickness d and as well is sufficiently small when compared to L_1/m and L_2/n, we can introduce in equations (5.5) the small parameter $\varepsilon = \beta_n l$.

The frequencies of free vibrations are calculated by assuming that the determinant of (5.5) is equal to zero. Denoting by

$$B_0 = \langle C_{11} h_{,1}^2 \rangle + \langle C h_{,1}^2 \rangle + \kappa^2 \left(\langle D_1 \rangle + \langle D_2 \rangle \right), \quad H_{11} = \langle C_{11} \rangle - \frac{\langle C_{11} h_{,1} \rangle^2}{B_0}$$

where the parameter $\kappa^2 = l^2/12 \cdot j^{-1} = \left(l/d \right)^2$ represent the influence of the length scale effect. Then solve the dispersion equation resulting from (5.5), taking into account that $\varepsilon \ll 1$, we arrive at the following approximate formulae for free vibrations frequencies :

lower

$$\omega_1^2 = j \frac{H_{11} \alpha_m^4 + 2\alpha_m^2 \beta_n^2 \left(\langle C_{12} \rangle + 2 \langle C \rangle \right) + \langle C_{22} \rangle \beta_n^4}{\langle \rho \rangle} + O\left(\varepsilon^6 \right) \tag{5.6}$$

and higher

$$\begin{aligned} \omega_2^2 &= \frac{\langle C_{11} h_{,1}^2 \rangle + \langle C h_{,1}^2 \rangle + \kappa^2 \left(\langle D_1 \rangle + \langle D_2 \rangle \right)}{2 j \kappa^2 \langle \rho \rangle} + O\left(\varepsilon^2 \right) \\ \omega_3^2 &= \frac{\langle D_1 \rangle}{j \langle \rho \rangle} + O\left(\varepsilon^2 \right) \\ \omega_4^2 &= \frac{\langle D_2 \rangle}{j \langle \rho \rangle} + O\left(\varepsilon^2 \right) \end{aligned} \tag{5.7}$$

Commenting the obtained results, notice that the vibration frequencies ω_3, ω_4 describe the effect of the plate rotational inertia on the dynamic behaviour. Neglecting in (5.2) terms $j \langle \rho \rangle \ddot{\vartheta}_\alpha^0$ we obtain only two basic free vibration frequencies. The lower frequency ω_1 can be compared with the earlier results obtained by using homogenization procedures. Also, to the similar formula for ω_1 we arrive in the framework of the elastic anisotropic plates [7]. The higher free vibration frequency ω_2 is caused by the plate uniperiodic structure, depends on the period l and cannot be derived from the homogenized model.

5.3. Numerical calculations

In this subsection the analysis of interrelation between non-dimensional lower free vibration frequency and geometrical parameters $\kappa = l/d$ and $\xi = L_2/L_1$ will be carried out.

Let us denote by C', C'' and D', D'' the elastic moduli of components of the periodicity cell, Fig.2. If $x = l'/l, x \in (0,1)$ the averaged operator reduces to the form

$$\langle f \rangle = x f' + (1 - x) f''$$

and

$$\langle f h^2 \rangle = l^2/12 \langle f \rangle, \langle f h_{,1} \rangle = f' - f'', \langle f h_{,1}^2 \rangle = f'/x + f''/(1 - x).$$

Therefore, for the uniperiodic, orthotropic plate under consideration we obtain

$$\langle C_{11}\rangle = xC_{11} + (1-x)C_{22}\ ,\ \langle C_{22}\rangle = xC_{22} + (1-x)C_{11}$$
$$\langle D_1\rangle = xD_1 + (1-x)D_2\ \ ,\ \langle D_2\rangle = xD_2 + (1-x)D_1$$
$$\langle \rho\rangle = \rho\ \ ,\ \langle C_{12}\rangle = C_{12}\ \ ,\ \langle C\rangle = C$$

Assuming that m=n=1 , multiplying relation (5.6) by $\rho\left(j\pi^4 C_{11}\right)^{-1} L_2^4$ we arrive at the following formulae for non-dimensional lower free vibration frequency

$$\Omega\ = \frac{1}{C_{11}}\left[\xi^4 \mathrm{H} + 2\xi^2\left(C_{12}+2C\right) + xC_{22} + (1-x)C_{11}\right] \tag{5.8}$$

where

$$H = \frac{\dfrac{1}{x(1-x)}\left\{C\left[xC_{11}+(1-x)C_{22}\right]+C_{11}C_{22}\right\}+\kappa^2\left(D_1+D_2\right)\left[xC_{11}+(1-x)C_{22}\right]}{\dfrac{1}{x(1-x)}\left[(1-x)C_{11}+xC_{22}+C\right]+\kappa^2\left(D_1+D_2\right)}$$

The calculation assumptions are fulfilled by glued timber plate that is composed of the elements cut along and across the fibres, Fig.3. According to *PN-B*-03150-2000, timber is a quasi-isotropic material with elastic moduli (with received denotation) : E_{11} = 13000 Mpa ,

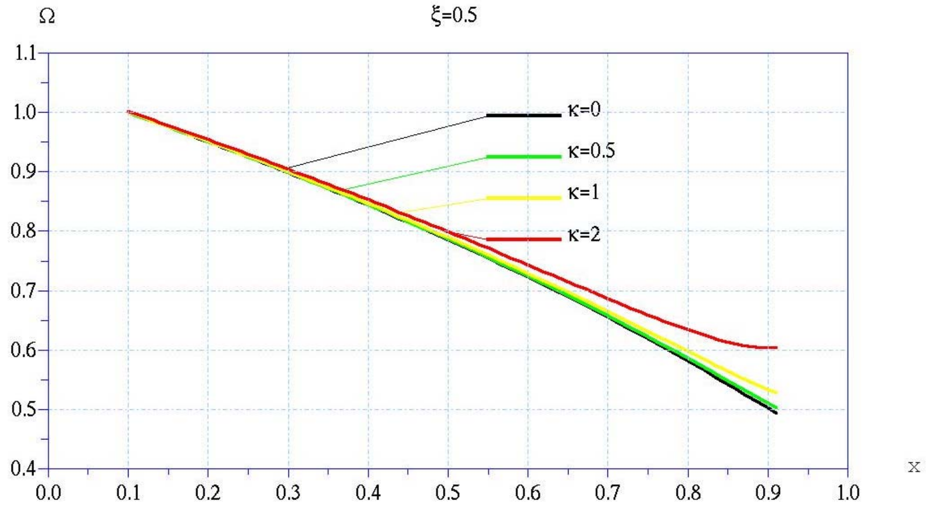
Fig. 3. An example of glued timber plate

E_{22} = 430 Mpa , D_1=D_2=C_{12}=C= 810 Mpa , for glued timber *GL*-35.

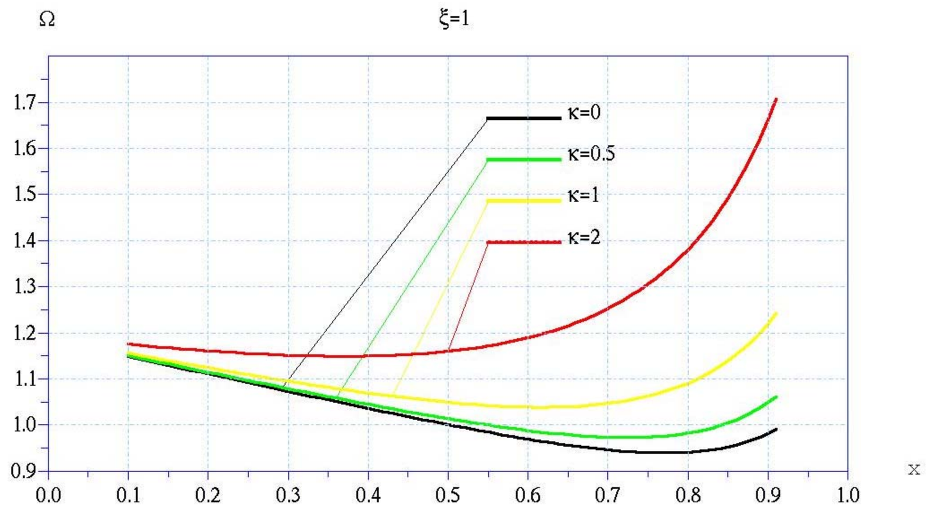
Calculations are performed for three values parameter ξ = 0,5 ; 1,0 ; 2,0 and for four values of κ = 0, ; 0,5 ; 1,0 ; 2,0 . For $\kappa = 0$, the obtained results one can treated as certain approximation of the homogenized model.. Diagrams representing interrelation between frequency Ω and the size of the periodicity cell(includes in x and κ) as well the parameter ξ , for glued timber plate, are shown in Fig.4.

Fig. 4.Interrelation between free vibrations frequency and geometrical parameters for glued timber plate

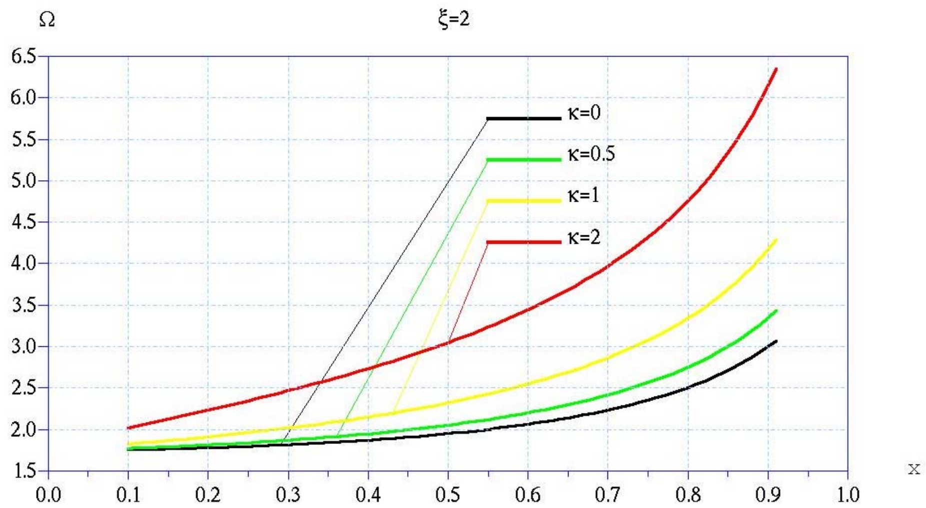
a)



b)



c)



Commenting the obtained results it should be stated that, with the adapted assumption regardless of glued timber plate, the asymptotic model gives the lowest values of the free vibrations frequency. The influence of in-homogeneity on the vibration frequency is considerably higher for plates in which the "periodic" length dimension L_1 (Fig.1.) is of an order L_2 or smaller. For given ξ , the frequency values rises with the growth of κ . If κ is small, there are no significant differences in the values of vibrations frequencies for proposed and homogenized model.

6. CONCLUSIONS

Summarizing the obtained results the following conclusions can be formulated.

- The proposed 2D-model of uniperiodic, Reissner–type plates makes it possible to investigate dynamic and stability problems, in which the constant plate thickness *d* is of an order of the period *l.*
- This model is derived by using the tolerance averaging technique, describes the effect of the period length *l* on the overall plate behaviour. Contrary to the homogenized model, the model obtained in this contribution, determines also higher free vibrations frequencies, caused by the plate uniperiodic structure.
- The obtained 2D-model is a certain complementation and extension for the model presented in [1], [2], [3] where the period length is assumed to be much larger when compared to the plate thickness.
- The analysis in section 5 but also in [15] confirms the thesis, that if the period length is small when compared to the plate thickness, then the length-scale effect is reduced; in this case the homogenization approach is used.
- The proposed model cannot be treated as a special case of nonasymptotic model of plates with bi-directional periodic structure [15], in spite of many convergences in the modelling procedure.
- The calculations for rectangular, simply supported glued timber plate lead to the conclusion, that the asymptotic model gives lower values of the basis free vibrations frequency. The influence of the period length $l=l_1$ on the frequency values rises with the growth of quotient L_2/L_1 .

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