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A CONTRIBUTION TO THE MODELLING OF HEAT CONDUCTION IN PERIODICALLY MULTILAYERED LAMINATES

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ABSTRACT

The subject of contribution is non-stationary heat conduction in a periodically inhomogeneous rigid laminated conductor. As a tool of modelling the tolerance averaging technique is taken into account, [1]. The aim of the contribution is to extend the tolerance averaged model of the heat conduction in periodically two-constituent laminates for the laminates with an arbitrary number of constituents. The approach is based on applying FEM for periodic structures in the framework of model with the micro-local parameters, [2]. Similar problem for the case without scale effect was examined in [3]. Related modelling problems was also discussed in [4,5,6,7].

Key words: heat conduction, periodically inhomogeneous conductor, temperature fluctuation evolution.

INTRODUCTION

The governing equations of such structures have highly oscillating and non-continuous functional coefficients. The direct numerical solutions to this equation is an ill-conditioned and complicated computational problems. That is why many authors propose and apply various approximated macroscopic models of the periodic solids and structures. For non-stationary problems in which the scale effect should be taken into account the tolerance averaging technique (TAT), developed in a series of papers and summarized in [2], plays an important role. In this paper the TAT will be applied to the formulation of a certain averaged model for heat conduction in a periodically inhomogeneous rigid laminated conductor. Considerations will be restricted to the simplest Fourier heat transfer law. As in most applications of TAT to the modelling of physical problems the formulation of the model reduces here to the construction of a certain finite system of shape functions. From a formal point of view the modelling approach proposed in this paper is identical to that presented in [1] for the two-constituent laminated conductors. For two-constituent laminates introduced system of shape functions reduces to the case of one-shape function proportional to the well known saw-like function which was successfully applied to the modelling of many problems for periodic material structures.

In the paper the attention is focused on the modelling of non-stationary problems for laminated rigid conductors with periodic microstructure. A similar problem, for the case in which the scale effect does not have to be taken into account, was discussed by Matysiak and Woźniak in [3]. However, the idea of the approach to the formulation of finite system of shape functions for a laminated medium applied in the aforementioned paper is quite different than that proposed in this paper.

Throughout the paper n denotes a number of constituents of the considered laminated conductor and N denotes the number of shape functions in the related tolerance model. Similarly as in [1] capital superscripts $A,B,\ldots$ are related to the enumeration of shape functions and run over the sequences 1,2,...,N. Lower superscripts $a,b,\ldots$ run over the sequences 1,2,...,n and deal to the enumeration of the constituents of the conductor. Summation convention holds provided that it will be stated otherwise. Gradients with respect to the space coordinates are denoted by ∇ and time derivatives by the overdot. Moreover denotations $\partial \equiv \partial/\partial x_3$ and $\overline{\nabla} \equiv (\partial/\partial x_1, \partial/\partial x_2, 0)$ will be applied.

FUNDAMENTAL CONCEPTS AND ASSUMPTIONS

To make this paper self-consistent, in the subsequent section basic ideas and concepts related to the application of tolerance averaging technique of PDEs with highly oscillating periodic coefficients are explained. For details the reader is referred to [1]. We consider heat conduction in a periodically multi-constituent laminate. The specific heat field and conduction tensor field will be denoted by $c(\cdot)$ and $\mathbf{K}(\cdot)$, respectively. We shall introduce the Carthesian orthogonal coordinate system $Ox_1x_2x_3$ and assume that periodicity direction is determined by Ox_3–axis while in every direction normal to this axis the properties of the medium are constant. Moreover, we shall assume that the region occupied by the conductor will be identified with $\Omega \equiv \Pi \times (-L, L)$, where Π is a certain region in the Ox_1x_2-plane. Denoting by n the number of constituents of the laminated medium under consideration fields $c(\cdot)$ and $\mathbf{K}(\cdot)$ we restrict to the form

$$\begin{aligned} c(x_3) &= c^1\chi^1(x_3) + c^2\chi^2(x_3) + \ldots + c^N\chi^N(x_3) \\ \mathbf{K}(x_3) &= \mathbf{K}^1\chi^1(x_3) + \mathbf{K}^2\chi^2(x_3) + \ldots + \mathbf{K}^N\chi^N(x_3) \end{aligned} \tag{1}$$

in which $\chi^a(\cdot)$, a=1,...,n, denote l-periodic characteristic functions of subsets of $(-L, L)$ occupied the a-th material component of this medium. The length l will be referred to as *a microstructure length parameter* and we shall assume that it is sufficiently small when compared to the minimum characteristic length dimension L of a region Ω in the Ox_3–axis direction. Moreover, a special form

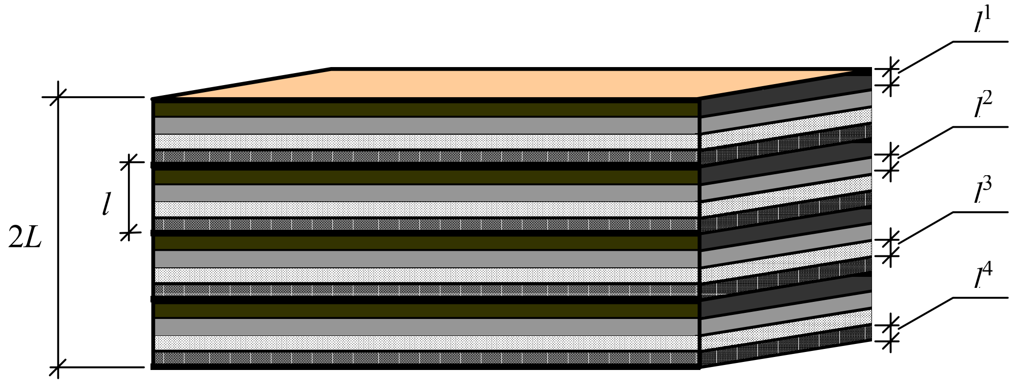
$$\mathbf{K} \equiv \begin{bmatrix} \overline{\mathbf{K}} & 0 \\ 0 & k \end{bmatrix} \tag{2}$$

of the conductivity tensor **K** will be applied. In the above form of conductivity tensor $\overline{\mathbf{K}} = (\overline{K}_{\alpha\beta})$ and

$$\begin{aligned} k(x_3) &= k^1\chi^1(x_3) + k^2\chi^2(x_3) + \ldots + k^n\chi^n(x_3) \\ \overline{\mathbf{K}}(x_3) &= \overline{\mathbf{K}}^1\chi^1(x_3) + \overline{\mathbf{K}}^2\chi^2(x_3) + \ldots + \overline{\mathbf{K}}^n\chi^n(x_3) \end{aligned} \tag{3}$$

An example of the conductor under consideration for N=5 constituents is presented in Fig.1.

Fig.1. Multilayered *l*-periodic rigid conductor



In the subsequent considerations a crucial role plays the concept of slowly-varying function. Roughly speaking, differentiable real valued function $F(\cdot)$ defined in [-*L*,*L*] is called *slowly-varying function* (with respect to the length *l* and some positive tolerance parameter ε , ε <<1) provided that condition

$$lF'(z)\in \varepsilon O(F(z)) \tag{4}$$

holds for every z∈ [-*L*,*L*]. Subsequently, slowly varying function F can also depend on x_1, x_2 and time *t* as parameters. If *F* is slowly varying together with all their derivatives which occur in manipulations then we shall write

$$F(x_1,x_2,\cdot,t)\in SV_\varepsilon(l) \tag{5}$$

For every integrable function $f(\cdot)$ the averaging operator will be defined by

$$\langle f(y)\rangle(z)\equiv\frac{1}{l}\int_{z-l/2}^{z+l/2} f(z+y)dy \tag{6}$$

It follows that for every $F\in SV_\varepsilon(l)$ and every $f,g\in L^\infty_{per}(0,l)$ conditions

$$\begin{aligned}&\langle fF\rangle(z)=\langle f\rangle F(z)+O(\varepsilon)\\&\langle f\partial(gF)\rangle(z)=\langle f\partial g\rangle F(z)+O(\varepsilon)\end{aligned} \tag{7}$$

hold. We shall assume that the condition usually called *tolerance averaging condition* is satisfied. This condition states that in the course of modelling in formulas (7) terms $O(\varepsilon)$ can be omitted.

Macroscopic model for the heat transfer in a rigid conductor with periodic microstructure is based on the tolerance averaging condition and two modelling assumptions. *The tolerance averaging condition* states that in the course of modelling in formulas (7) terms $O(\varepsilon)$ can be omitted.

In order to formulate mentioned above modelling assumptions we denote by θ (**x**,*z*,*t*), $z=x_3$, **x**∈ Π the temperature field at time *t* and introduce the averaged temperature by

$$\vartheta(\mathbf{x},z,t)\equiv\langle c\rangle^{-1}\langle c\theta\rangle(\mathbf{x},z,t) \tag{8}$$

The first modelling assumption restricts considerations to problems in which averaged temperature field for every *t* is a slowly varying function with respect to variable x_3 together with all their derivatives, i.e.

$$\vartheta(x_1,x_2,\cdot,t)\in SV_\varepsilon(l) \tag{9}$$

The second modelling assumption states that temperature fluctuations defined by $\theta - \vartheta$ can be approximated by the formula

$$\theta(\mathbf{x}, z, t) - \vartheta(\mathbf{x}, z, t) \cong g^A(z)\psi^A(\mathbf{x}, z, t) \tag{10}$$

where $g^A(\cdot)$, A=1,…,N, is a system of linear independent functions called *shape functions* and $\psi^A(x_1, x_2, \cdot, t)$, A=1,…,N, are slowly-varying functions together with all their derivatives, i.e.

$$\psi^A(x_1, x_2, \cdot, t) \in SV_\varepsilon(l), \qquad A = 1, \ldots, N \tag{11}$$

Shape functions which are l-periodic continuous functions, have to satisfy conditions

$$\langle cg^A \rangle = 0 \tag{12}$$

The number N and the form of shape functions have to be postulated *a priori* in every special problem. Functions $\psi^A(x_1, x_2, \cdot, t)$ will be called *fluctuation amplitudes.* Bearing in mind the above modelling assumptions we conclude that

$$\theta(\mathbf{x}, z, t) \cong \vartheta(\mathbf{x}, z, t) + g^A(z)\psi^A(\mathbf{x}, z, t) \tag{13}$$

In the subsequent analysis averaged temperature ϑ and fluctuation amplitudes ψ^A are basic unknowns. Governing equations for these unknowns will be derived in the subsequent section.

MODEL EQUATIONS

The starting point of the modelling procedure is the well-known Fourier heat conduction equation

$$c\dot{\theta} - \partial(k\partial\theta) - \overline{\nabla} \cdot (\overline{\mathbf{K}} \cdot \overline{\nabla}\theta) = 0 \tag{14}$$

Applying the averaging operator (6) to the left hand side of the above equation and bearing in mind (13) we obtain

$$\langle c \rangle \dot{\vartheta} - \langle k \rangle \partial\partial\vartheta - \langle k\partial g^A \rangle \partial\psi^A - \langle \overline{\mathbf{K}} \rangle : \overline{\nabla}\overline{\nabla}\vartheta + \langle \overline{\mathbf{K}} g^A \rangle : \overline{\nabla}\overline{\nabla}\psi^A = 0 \tag{15}$$

Similarly, multiplying by ψ^A both sides of equation (14), averaging equation obtained on this way and bearing in mind (13) we arrive at the system of model equations

$$\begin{aligned} &\langle cg^A g^B \rangle \dot{\psi}^B - \langle k\partial g^A \partial g^B \rangle \psi^B - \langle \overline{\mathbf{K}} g^A g^B \rangle : \overline{\nabla}\overline{\nabla}\psi^B - \\ &- \langle kg^A \partial g^B \rangle \partial\psi^B - \langle kg^A \rangle \partial\partial\vartheta + \langle k\partial g^A \rangle \partial\vartheta - \langle \overline{\mathbf{K}} g^A \rangle : \overline{\nabla}\overline{\nabla}\vartheta = 0 \end{aligned} \tag{16}$$

For the detailed analysis of the above model equations the reader is referred to [2]. Now we shall restrict considerations to the case in which conditions

$$\begin{aligned} &k(x_3) = \mu c(x_3), \quad x_3 \in [-L, L] \\ &K_{\alpha\beta}(\cdot) = \mu_{\alpha\beta} k(x_3), \quad \mu, \mu_{\alpha\beta} - const. \end{aligned} \tag{17}$$

which will be referred to as the *proportional heat conduction assumptions*, hold. In this case coefficients $\langle \overline{\mathbf{K}} g^A \rangle$, $\langle kg^A \rangle$ vanish and system equations (15)and (16) can be rewritten in the final form

$$\begin{aligned} &\langle c \rangle \dot{\vartheta} - \langle k \rangle \partial\partial\vartheta - \langle \overline{\mathbf{K}} \rangle : \overline{\nabla}\overline{\nabla}\vartheta - \langle k\partial g^A \rangle \partial\psi^A = 0 \\ &\langle cg^A g^B \rangle \dot{\psi}^B - \langle \overline{\mathbf{K}} g^A g^B \rangle : \overline{\nabla}\overline{\nabla}\psi^B - \langle k\partial g^A \partial g^B \rangle \psi^B - \\ &- \langle kg^A \partial g^B \rangle \partial\psi^B + \langle k\partial g^A \rangle \partial\vartheta = 0 \end{aligned} \tag{18}$$

representing tolerance model equations for a heat conduction in periodically multi-layered rigid conductor for which the *proportional heat conduction assumptions* are satisfied.

SHAPE FUNCTIONS

The aim of this paper is to formulate and analyze a certain system of shape functions for the tolerance averaged model of multi-layered rigid conductors under consideration. This procedure will be realized in five steps.

***Step* 1**. *Introducing an infinite partition of the interval* $\Delta \equiv [0,l]$.

Let us introduce the infinite increasing sequence of real numbers $0<y^1<\ldots<y^n=l$ which will be termed as *nodes*. Every such sequence determines a finite partition of the interval $\Delta \equiv [0,l]$ onto n subintervals $\Delta_a \equiv \left(y^a - l^a, y^a\right), a = 1,...,n$, where l^a denotes a length of a-th interval Δ_a. Hence $cl\Delta = \bigcup_{a=1}^{n} cl\Delta_a$, where cl is the closure operation in the real axis and $\Delta_a \cap \Delta_b = \varnothing$ for $a \neq b$, $a,b=1,\ldots,n$. Hence, the introduced family of intervals can be treated as a finite partition of the interval Δ. We shall also denote $\eta^a \equiv l^a / l$ for $a = 1,...,n$.

***Step* 2**. *Introducing a system of interpolation functions*.

To every node y^a, $a=1,\ldots n$, the continuous real function ϕ_a supported on $\Delta_a \cup \{y^a\} \cup \Delta_{a+1}$ will be assigned. Functions ϕ_a will be defined by conditions

1) $\phi_a\left(y^b\right) = \delta_a^b$,
2) $\phi_a\left(y\right) = 0$ for $y \notin \left(\Delta_a \cup \left\{y^a\right\} \cup \Delta_{a+1}\right)$
3) $\phi_a\left(\cdot\right)$ are linear in every Δ_b, $b=1,\ldots,n$

in which intervals $\Delta_1 \equiv (y^0, y^1)$ and $\Delta_{n+1} \equiv (y^n, y^n + l^1)$ are treated as the same. Functions ϕ_a will be called *interpolation functions*.

***Step* 3**. *Introducing a system of basis functions*.

The system of *basis functions* h^a, $a=1,..,n$, we are to define in this step, is a finite sequence of real functions defined on the interval Δ=[0,l] and strictly connected with the system of l-periodic shape functions g^A, $A=1,..,N$, which plays a crucial role in the formulation of tolerance averaged model of the laminated conductor under consideration and will be defined in the next step. The number n of basis functions is identical with the number of different nodes in the interval Δ=[0,l]. The a-th function h^a, $a=1,..,n$, from the system of *basis functions* will be identified with

$$h^a = \phi_a - \gamma^a \phi_{a+1} \text{ (no summation over } A!) \tag{19}$$

where constants γ^A which can be determined from by the condition $\langle ch^A \rangle = 0$, i.e.

$$\frac{\eta^a \langle ch^a \rangle_a + \eta^{a+1} \langle ch^a \rangle_{a+1} + \eta^{a+2} \langle ch^a \rangle_{a+2}}{\eta^a + \eta^{a+1} + \eta^{a+2}} = 0 \quad \text{(no summation over } A!) \tag{20}$$

In the above formulas summation in the set of indices which enumerate the set of nodes should be understand in the well known sense modulo n and where for every integer $a=1,\ldots,n$ symbol

$$\langle f \rangle_a \equiv \frac{1}{l^a} \int_{\Delta_a} f(z) dz \tag{21}$$

denotes the averaged value on the interval Δ_a of an integrable function *f*. After simple calculation from condition (20) we obtain

$$\gamma^a = \frac{c^a \eta^a + c^{a+1} \eta^{a+1}}{c^{a+1} \eta^{a+1} + c^{a+2} \eta^{a+2}} \text{ (no summation over } A!) \tag{22}$$

To the system of basis functions h^a and to an arbitrary function $v(\cdot)$ determined by the sequence of positive constants $v^1,...,v^N$ by the formula

$$v(x_3) = v^1 \chi^1(x_3) + v^2 \chi^2(x_3) + ... + v^N \chi^N(x_3) \tag{23}$$

a certain system of constants will be assigned. To this end we define

$$\begin{aligned}
\langle v(h^A)' \rangle_N &\equiv \sum_{a \in I_N^A} \eta^a v^a \langle (h^A)' \rangle_a \\
\langle v(h^A)'(h^B)' \rangle_N &\equiv \sum_{a \in I_N^{AB}} \eta^a v^a \langle (h^A)'(h^{A+1})' \rangle_a \\
\langle v h^A h^B \rangle_N &\equiv \sum_{a \in I_N^{AB}} \eta^a v^a \langle h^A h^{A+1} \rangle_a \\
\langle v h^A (h^B)' \rangle_N &\equiv \sum_{a \in I_N^{AB}} \eta^a v^a \langle h^A (h^{A+1})' \rangle_a
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
I_N^A &\equiv \{A-1, A, A+1\} \cap \{1,...,N\} (\operatorname{mod} N) \\
I_N^{AB} &\equiv \{A-1, A, A+1\} \cap \{1,...,N\} \cap \{B-1, B, B+1\} (\operatorname{mod} N)
\end{aligned} \tag{25}$$

***Step* 4**. *Introducing a final form of shape functions for periodic structure*

Unfortunately, *n* basis functions defined in the previous step are not independent. However, it is easy to verify that every *n*-1 functions chosen from them are independent. To define the system of *l*-periodic shape functions g^A we shall firstly reduce the number of basis functions by dropping out one from them. Secondly we are to transform every basis functions h^a to the certain functions of an order $O(l)$. Finally we shall extend functions obtained on this way to *l*-periodic functions.

To obtain the system of independent basis functions the *n*-th function from this system will be dropped out. It is mean that the interface between *n*-th and *n*+1-th laminae is distinguished. To satisfy condition condition (12) we shall replace every basis function h^a by the function $l\sqrt{\eta^a \eta^{a+1}} h^a$ (no summation over *a*!). Now the restriction of *A*-th shape function g^A to the interval Δ=[0,*l*] shall be defined as

$$g^A |_\Delta \equiv l\sqrt{\eta^A \eta^{A+1}} h^A \tag{26}$$

where *N*=*n*-1. Above condition uniquely determines *l*-periodic shape functions g^A *A*=1,…,*N*. Now coefficients in model equations (18) are equal to (no summation over *A*!):

$$
\begin{aligned}
&\langle k(g^A)'\rangle \equiv l\sqrt{\eta^{A-1}\eta^A}\,\langle k(h^A)'\rangle_N \\
&\langle k(g^A)'(g^B)'\rangle \equiv l^2\sqrt{\eta^{A-1}\eta^{B-1}\eta^A\eta^B}\,\langle k(h^A)'(h^B)'\rangle_N \\
&\langle cg^Ag^B\rangle \equiv l^2\sqrt{\eta^{A-1}\eta^{B-1}\eta^A\eta^B}\,\langle ch^Ah^B\rangle_N \\
&\langle kg^A(g^B)'\rangle \equiv l^2\sqrt{\eta^{A-1}\eta^{B-1}\eta^A\eta^B}\,\{kh^A(h^B)'\rangle_N
\end{aligned}
\qquad (27)
$$

Summarizing considerations of this section it must be emphasized that coefficients (30)-(33) as well as coefficients (28) are not a special cases of coefficients (34)-(37) for a simple valuation of the number *N*. Moreover, for *N*=2 term in the model equation (18) related to coefficient $(28)_4$ vanishes.

SPECIAL CASES

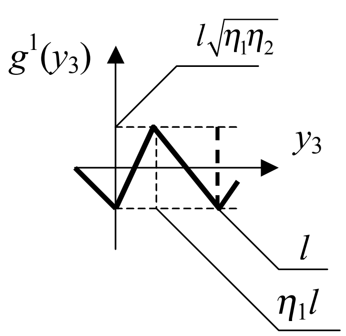
Now we are to calculate coefficients (24). The new forms of these coefficients shall include exclusively constants η^A, ν^A for *A*=1,…,*N* and constants γ^A for *A*=1,…,*N*-1. To this end we are to examine three special cases.

As the first case let us consider *N*=2. Hence, we deal with two-constituent periodically layered laminate and indices *A* run over exclusively one integer equal to one. From (22) we obtain only one coefficient $\gamma=\gamma^A$ which takes the value $\gamma=1$ and the sequence of periodic basis function defined by (19) reduces to exclusively one basis function denoted here by *h* and illustrated in the Fig. 2. In the case under consideration coefficients (24) take values

$$
\begin{aligned}
&\langle kh'\rangle = 2l^{-1}(k_1-k_2) \\
&\langle k(h')^2\rangle = 4l^{-2}\left(\frac{k_1}{\eta_1}+\frac{k_2}{\eta_2}\right) \\
&\langle chh\rangle = \frac{1}{3}\langle c\rangle = \frac{1}{3}(\eta_1c_1+\eta_2c_2) \\
&\langle kh(h)'\rangle = 0
\end{aligned}
\qquad (28)
$$

Now, if we define $\overline{g}\equiv l\sqrt{3}h$ we obtain well-known saw-lake function usually applied in the considerations dealing two-phase laminates, cf. [2].

Fig. 2. Periodic basis function for the two-constituent periodic rigid conductor



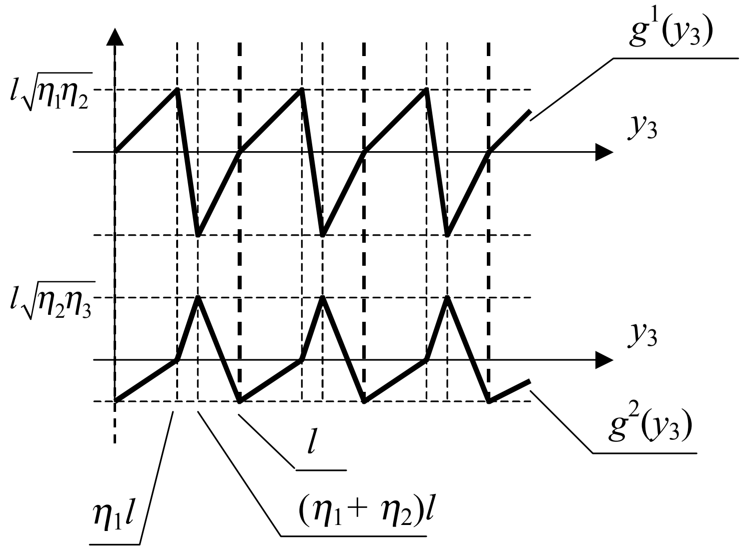
As a second case we consider *N*=3. In this case we deal with two basis functions illustrated in Fig. 3. From (22) we obtain two coefficients γ^1 and γ^2 equal to

$$\gamma^1 = \frac{c_1\eta_1 + c_2\eta_2}{c_2\eta_2 + c_3\eta_3}, \qquad \gamma^2 = \frac{c_2\eta_2 + c_3\eta_3}{c_2\eta_2 + c_3\eta_3} \tag{29}$$

In this case one can obtain one column matrix for coefficient $(24)_1$:

$$[\langle k(h^A)'\rangle] = 2l^{-1}[k_1 - (1+\gamma^1)k_2 + \gamma^1 k_3, k_2 - (1+\gamma^2)k_3 + \gamma^2 k_1]^T \tag{30}$$

Fig. 3. Periodic basis function for the three-constituent periodic rigid conductor



For the second coefficient $(24)_2$ in the case under consideration we obtain

$$[\langle k(h^A)'(h^B)'\rangle] = l^{-2}\begin{bmatrix} \frac{k_1}{\eta_1} + \frac{(1+\gamma^1)^2 k_2}{\eta_2} + \frac{(\gamma^1)^2 k_3}{\eta_3} & \frac{\gamma^2 k_1}{\eta_1} - \frac{\gamma^2(1+\gamma^1)k_2}{\eta_2} - \frac{\gamma^1(1+\gamma^2)k_3}{\eta_3} \\ \frac{\gamma^2 k_1}{\eta_1} - \frac{\gamma^2(1+\gamma^1)k_2}{\eta_2} - \frac{\gamma^1(1+\gamma^2)k_3}{\eta_3} & \frac{k_2}{\eta_2} + \frac{(1+\gamma^2)^2 k_3}{\eta_3} + \frac{(\gamma^2)^2 k_1}{\eta_1} \end{bmatrix} \tag{31}$$

For the third coefficient $(24)_3$ in the case under consideration we obtain

$$\langle ch^A h^B\rangle = \begin{bmatrix} \frac{\eta_1 c_1}{3} + \frac{(1-\gamma_1+\gamma_1^2)\eta_2 c_2}{3} + \frac{\gamma_1^2\eta_3 c_3}{3} & -\frac{\gamma_2\eta_1 c_1}{6} + \frac{(1-2\gamma_1)\eta_2 c_2}{6} + \frac{(\gamma_2-2)\gamma_1\eta_3 c_3}{6} \\ -\frac{\gamma_2\eta_1 c_1}{6} + \frac{(1-2\gamma_1)\eta_2 c_2}{6} + \frac{(\gamma_2-2)\gamma_1\eta_3 c_3}{6} & \frac{\gamma_2^2\eta_1 c_1}{3} + \frac{\eta_2 c_2}{3} + \frac{(1-\gamma_2+\gamma_2^2)\eta_3 c_3}{3} \end{bmatrix} \tag{32}$$

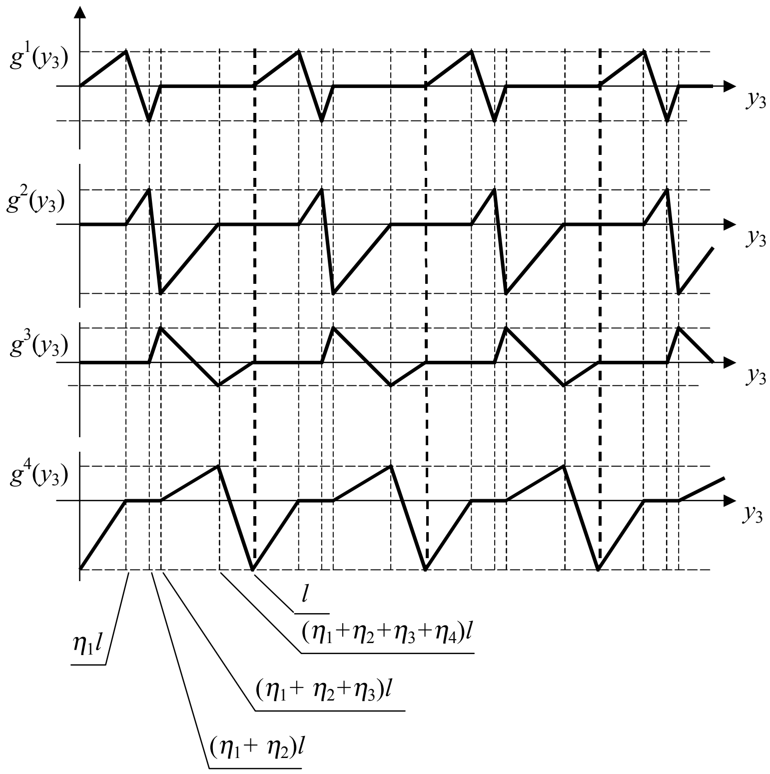
For the fourth coefficient $(24)_4$ in the case under consideration we obtain

$$\{kh^A(h^B)'\} = l^{-1}\begin{bmatrix} k_1 - \gamma_1(1+\gamma_1)k_2 + \gamma_1^2 k_3 & \gamma_2 k_1 - \gamma_1 k_2 + \gamma_1(1+\gamma_2)k_3 \\ \gamma_2 k_1 - \gamma_1 k_2 + \gamma_1(1+\gamma_2)k_3 & k_2 - \gamma_2(1+\gamma_2)k_3 + \gamma_2^2 k_1 \end{bmatrix} \tag{33}$$

Similar representations for coefficients (24) can be found in the last case.

The third case will be determined by $N>3$. In this case we deal with N-1 basis functions, which for N=5 are illustrated in Fig. 4.

Fig. 4. Periodic basis function for the five-constituent periodic rigid conductor



After simple calculations made for $\langle \nu (h^A)' \rangle_N$ one can obtain

$$\langle \nu (h^A)' \rangle_N = l^{-1}[\nu^A - (1+\gamma^A)\nu^A + \gamma^A \nu^{A+2}] \tag{34}$$

For $\langle \nu (h^A)'(h^B)' \rangle_N$ calculations lead to:

$$\langle \nu (h^A)'(h^B)' \rangle_N = l^{-2} \begin{cases} \dfrac{1}{\eta^A}\nu^A + \dfrac{(1+\gamma^A)^2}{\eta^{A+1}}\nu^{A+1} + \dfrac{(\gamma^A)^2}{\eta^{A+2}}\nu^{A+2} & for \quad A = B \\ -\dfrac{(1+\gamma^A)}{\eta^{A+1}}\nu^{A+1} - \dfrac{\gamma^A(1+\gamma^{A+1})}{\eta^{A+2}}\nu^{A+2} & for \quad B - A = 1 \\ \dfrac{\gamma^A}{\eta^{A+2}}\nu^{A+2} & for \quad B - A = 2 \\ 0 & if \;\; otherwise \end{cases} \tag{35}$$

For $\langle \nu h^A h^B \rangle_N$ we arrive at

$$\langle \nu h^A h^B \rangle_N = \begin{cases} \frac{-\gamma^{A+1}}{6}\eta^A \nu^A + \frac{1-2\gamma^A}{6}\eta^{A+1}\nu^{A+1} + \frac{(\gamma^{A+1}-2)\gamma^A}{6}\eta^{A+2}\nu^{A+2} & for \quad A=B \\ \frac{1-\gamma^A+(\gamma^A)^2}{6}\eta^{A+1}\nu^{A+1} + \frac{(\gamma^A)^2}{3}\eta^{A+2}\nu^{A+2} & for \quad B-A=1 \\ \frac{\gamma^{A+2}}{3}\eta^{A+2}\nu^{A+2} & for \quad B-A=2 \\ 0 & if \quad otherwise \end{cases} \tag{36}$$

Similarly, for $\{\nu h^A (h^B)'\}$ we obtain

$$\langle \nu h^A (h^B)' \rangle_N = $$
$$= l^{-1} \begin{cases} \frac{1}{2}[\nu^A - (1-\gamma_A{}^2)\nu^{A+1} - \gamma_A{}^2\nu^{A+2} & for \quad A=B \\ \frac{1}{2}[-(1-\gamma_A)\nu^{A+1} - (1+\gamma_{A+1})\gamma_A\nu^{A+2}] & for \quad B-A=1 \\ -\frac{1}{2}\gamma_A{}^2\nu^{A+2} & for \quad B-A=2 \\ 0 & if \quad otherwise \end{cases} \tag{37}$$

In the next section we are to discuss model equations in which shape functions are taken in the form (26)of just defined shape functions g^A, A=1,…,N-1.

DISCUSSION OF THE MODEL

In this section we are to prove that in the framework of the proposed model in the first approximation component of the heat flux vector in the direction perpendicular to the layering can be approximated with the accuracy to the term of an order $O(l)$ by a certain continuous function provided that temperature field is defined by the formula (13). To this end define

$$\psi_h{}^A = \frac{\psi^A}{l\sqrt{\eta^A \eta^{A+1}}} \tag{38}$$

and rewrite model equations (18) in the form

$$\begin{aligned} &\langle c \rangle \dot{\vartheta} - \langle k \rangle \partial\partial\vartheta - \langle \bar{\mathbf{K}} \rangle : \bar{\nabla}\bar{\nabla}\vartheta - \langle k\partial h^A \rangle \partial \psi_h{}^A = 0 \\ &l^2 \left(\langle c h^A h^B \rangle \dot{\psi}_h{}^B - \langle \bar{\mathbf{K}} h^A h^B \rangle : \bar{\nabla}\bar{\nabla}\psi_h{}^B \right) - \langle k\partial h^A \partial h^B \rangle \psi_h{}^B - \\ &-l \langle k h^A \partial h^B \rangle \partial \psi_h{}^B + \langle k \partial h^A \rangle \partial\vartheta = 0 \end{aligned} \tag{39}$$

We shall observe that

$$\det[\langle \partial h^A k \partial h^B \rangle_N] = (1+\gamma_1+\gamma_1\gamma_2+...+\gamma_1\gamma_2...\gamma_N)\frac{k_1\eta_1+k_2\eta_2+...+k_N k_N}{\eta_1\eta_2..\eta_N} \tag{40}$$

From the condition $\det[\langle \partial h^A k \partial h^B \rangle] \neq 0$ we deduce that $[\langle \partial h^A k \partial h^B \rangle]$ is nonsingular NxN matrix. Hence equation $(18)_2$ can be rewritten in the form

$$\psi^B = -H_N{}^{AB} \langle k \partial h^B \rangle \partial\vartheta + r^A(\vartheta, \psi^B) \tag{41}$$

where

$$r^A(\vartheta,\psi^B) \equiv -l^2 H_N{}^{AB}(\langle ch^B h^C\rangle \dot{\psi}^C + \langle \bar{\mathbf{K}} h^B h^C\rangle : \bar{\nabla}\bar{\nabla}\psi^C) - \\ - lH_N{}^{AB}\langle kh^B \partial h^C\rangle \partial\psi^C) \tag{42}$$

and $\mathbf{H}_N=(H_N{}^{AB})$ is a matrix inverse to $\langle k\partial h^A \partial h^B\rangle_N$. Let us determine from the second of the model equations (18) total temperature field

$$\theta = \bar{\theta} + l\sqrt{\eta^A \eta^{A+1}} h^A r^A(\vartheta,\psi^B) \tag{43}$$

where

$$\bar{\theta} = \theta^0 (1 - h^A H_N{}^{AB}\langle k\partial h^B\rangle) \tag{44}$$

Since $l\sqrt{\eta^A \eta^{A+1}} h^A \in O(l)$ for A=1,…,N-1 the term $l\sqrt{\eta^A \eta^{A+1}} h^A\ r^A(\vartheta,\psi^B)$ in (43) is small when compared with $\bar{\theta}$ and in many problems can be drop out. This situation is related to the quasi-stationary case in which in the model equation $(18)_2$ the term with the time derivative can be omitted. Now the heat flux in the direction perpendicular to the layering can be written in the form

$$\Phi(\mathbf{x},z) = k\partial\theta^o(z)[1 - h^A(z) H_N{}^{AB}\langle k\partial h^B\rangle], \quad A,B = 1,...,N \tag{45}$$

where $\mathbf{H}_N=(H_N{}^{AB})$ is a matrix inverse to $\langle \partial h^A k \partial h^B\rangle_N$. The jump of the heat flux on the interface located between A+1-th and A-th laminate in the direction perpendicular to the layering will be identified with the formula

$$[\Phi]^A = \partial\theta^o(\eta^A)\{k^{A+1} - k^{A+1} - (k^{A+1}h^B(\eta^A +) - k^A h^B(\eta^A -))H^{BC}\langle k\partial h^C\rangle\}, \\ A = 1,...,N+1 \tag{46}$$

and will be referred to as *A-th jump of the heat flux* and we introduce vector $[\mathbf{\Phi}] \in R^{N+1}$ defined as

$$[\mathbf{\Phi}] = ([\Phi]^1, [\Phi]^2, ..., [\Phi]^N, [\Phi]^{N+1})^T \tag{47}$$

which will be called *the heat flux jump vector*. We are to prove that this vector is equal to zero. Similar fact for two-layered laminated rigid conductor and quasi-stationary case has been proved in [3]. Let us introduce the matrix **M**:

$$M_N{}^{AB} \equiv k^{A+1} h^B(\eta^A +) - k^A h^B(\eta^A -), \quad A = 1,...,N \tag{48}$$

and the vector, which B-th element are equal to

$$sr(\mathbf{M}_N)^B \equiv \sum_{A=1}^{N} M_N{}^{AB} \tag{49}$$

Under the above denotations the jump of the heat flux can be rewritten in the form

$$[\mathbf{\Phi}]^A = \partial\theta^o(\eta^A)\{k^{A+1} - k^{A+1} - \begin{bmatrix} M_N{}^{AB} \\ sr(\mathbf{M}_N)^B \end{bmatrix} H^{BC}\langle k\partial h^C\rangle\} \tag{50}$$

and thus *the heat flux jump vector* $[\mathbf{\Phi}]$ can

$$[\mathbf{\Phi}] = \partial\theta^o(\eta^A)\{\Delta\mathbf{k} - \begin{bmatrix} \mathbf{M}_N \\ sr(\mathbf{M}_N) \end{bmatrix} \cdot \mathbf{H}_N \cdot \langle k\partial\mathbf{h}\rangle\} \tag{51}$$

where

$$\langle k\partial\mathbf{h}\rangle \equiv (\langle k\partial h^1\rangle, \langle k\partial h^2\rangle, ..., \langle k\partial h^N\rangle)^T \tag{52}$$

Simple calculation yields

$$\langle k\partial\mathbf{h}\rangle \equiv [k_1 - (1+\gamma^1)k_2 + k_3, k_2 - (1+\gamma^2)k_2 + k_3, ..., k_N - (1+\gamma^1)k_{N+1} + k_{N+2}]^T \tag{53}$$

summation in (53) shall be understand in the modulo N sense.

Now we shall find a clear form of the matrix M. It is a crucial idea of the proof of heat flux continuity. To this end we shall introduce the sequence of matrix $\mathbf{N}_n$, which is defined for positive integer n by the following recurrence formula

$$\mathbf{N}_1 \equiv 1$$
$$\mathbf{N}_n \equiv \begin{bmatrix} 1 & \gamma^1 & ... & \gamma^1\gamma^2...\gamma^n & \gamma^1\gamma^2...\gamma^{n+1} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \gamma^n & \gamma^n\gamma^{n+1} \\ 0 & 0 & ... & 1 & \gamma^{n+1} \\ 0 & 0 & ... & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{N}_n & 0 \\ 0 & 0 \end{bmatrix} + \gamma^{n+1} \begin{bmatrix} 0 & 0 & ... & 0 & \gamma^1\gamma^2...\gamma^n \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & \gamma^n \\ 0 & 0 & ... & 1 & 1 \\ 0 & 0 & ... & 0 & 1 \end{bmatrix} \tag{54}$$

for every positive integer n. Hence

$$\mathbf{N}_n \equiv \begin{bmatrix} 1 & \gamma^1 & ... & \gamma^1\gamma^2...\gamma^n & \gamma^1\gamma^2...\gamma^{n+1} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & \gamma^n & \gamma^n\gamma^{n+1} \\ 0 & 0 & ... & 1 & \gamma^{n+1} \\ 0 & 0 & ... & 0 & 1 \end{bmatrix} \tag{55}$$

By $sr(\mathbf{N})$ we denote a one versus matrix, which B-th elements are equal to

$$sr(\mathbf{N})^B \equiv \sum_{A=1}^{N} N^{AB} \tag{56}$$

respectively. It is easy to verify the following recurrence formula

$$\mathbf{M}_1 = 1$$
$$\mathbf{M}_{n+1} = \gamma^{n+1} \begin{bmatrix} \mathbf{M}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{N}_n \\ -sr(\mathbf{N}_n) \end{bmatrix}, \qquad n = 1, 2, ... \tag{57}$$

Bearing in mind (53) we conclude that

$$\mathbf{M}_N \cdot \langle k\partial\mathbf{h}_N\rangle = [k_2 - k_1, k_3 - k_2, ..., k_{N+1} - k_N]^T \tag{58}$$

where indice N+1 shall be understand in the modulo N sense and under this assumption can be treated as equal to 1. Bearing in mind (51) we arrive at

$$[\Phi]^A = 0 \tag{59}$$

It means that in the framework of the proposed model component of the heat flux vector in the direction perpendicular to the layering is a certain continuous function. It must be emphasized that from the mathematical viewpoint this fact has been proved independently from the condition (20). It is mean that component of the heat flux vector in the direction perpendicular to the layering is a certain continuous function for an arbitrary sequence of constants γ^A, A=1,…,N-1.

FINAL REMARKS

The main results of this paper can be summarized as follows:

1. The finite system of shape functions for the non-stationary tolerance averaged model equations (18) of the heat conduction in the *n*-constituent periodically laminated rigid conductor were derived.
2. The form of constant coefficients in the above equations was calculated. The clear mathematical formula for the related effective heat conduction constant has been derived.
3. The continuity conditions of the heat flux across the interfaces between adjacent laminae interfaces were discussed. It was shown that for the stationary heat conduction problems the aforementioned continuous conditions are exactly satisfied. For non-stationary problems the above conditions are satisfied with an approximations of an order of the microstructure length parameter *l*.

Applications of results obtained in this paper will be explained in the separate note.

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