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COMPRESSION OF NON-STATIONARY SIGNALS BY THE WAVELET TRANSFORMATION

Siegfried Meier
Institut für Planetare Geodäsie, Technische Universität Dresden

Abstract

By means of the wavelet transformation not only the signals will be transferred into the wavelet domain, but also their statistical qualities, which are described by probability distributions, second order moment functions, and others. Here, we consider the normal distribution and the distributions of the ordinates of periodic signals in order to compare pre-estimates of compression rates using continuous wavelet transforms. The action of non-zero expectation values of processes with stationary increments is investigated for the same purpose. Finally, the theoretical results based on the continuous transformation are compared to those of numerical experiments with discrete data.

Key words: wavelet, compression rates

1 Introduction

The wavelet transformation (WT) is a linear integral transformation of the convolution type. Therefore it can be understood as a linear filter. Its weighting function g and its frequency response G (as to the notation of Louis, Maas and Rieder (1994) and with the exception of normalizing factors) are

$$g \sim \frac{1}{\sqrt{a}}\psi\left(-\frac{t}{a}\right), \qquad G \sim \sqrt{a}\hat{\psi}\left(-a\omega\right), \tag{1.1}$$

where ψ is the wavelet used, $\hat{\psi}$ is its Fourier transform (FT) and a is the scale parameter. Because of the properties of ψ this filter is of the band pass type: The WT of a signal f produces a band-filtered version of f on each scale $a > 0$. This is the basis of the decomposition of f into its components of different

Table 1: DTM Schneealpe, Austria. Empirical compression rates $\hat{k}_e$ using the discrete WT (according to G. Beyer). Discussion in Section 1, exampel 1.

Test	$\hat{\mu}$	$\hat{m}_h$ in m	S	$\hat{k}_e^{(1)}$	$\hat{k}_e^{(2,3,4,6)}$	$\hat{k}_e^{(2,3,4,6)}/\hat{k}_e^{(1)}$
1	0.27	2.2	6.6	3.9	6.0 bis 7.1	1.5 bis 1.8
2	0.52	2.8	7.9	2.4	5.8 bis 6.6	2.4 bis 2.8
3	0.67	3.1	10.0	2.2	5.8 bis 6.2	2.6 bis 2.8

"width", or analysis of "complex" signals. If the latter have statistical properties, these are transferred from the original domain into the wavelet domain (WD).

An empirical statistics with wavelet coefficients (WCs) is, at least on the denser occupied lower scales, always possible. But the one who is interested in the transformation mechanism knows that, on the one hand, narrow limitations are put on the analytical calculation already with signals and their mean values, even narrower limits when transforming second and higher moments, let alone the complete distribution functions. On the other hand, such transformations are important means to solve practical problems. Three areas of problems should be mentioned:

1. The fast WT provides thinned out series of WCs. In order to optimally interpolate them for approximation purposes, the auto-covariance functions (ACFs) of signal and noise in the WD are needed.

2. The mentioned (and other) characteristics can be used to analyse the properties of *one* input signal on *different* scales or of *two (several)* input signals on *one* scale. Schmidt (2000 to 2002) produced some theoretically based papers - from the continuous and discrete transformation of the moment functions up to test strategies, inclusive of the two-dimensional case (Schmidt 2001b).

3. The data compression in the WD is based on thresholding. To pre-estimate the number of the (negligible) WCs requires integration with respect to density functions of the WCs. In case of normally distributed WCs, apart from expectation values, the scale-dependent variances are needed.

In the following the latter problem is treated from special points of view. Previous investigations of the author (Meier 2003) were confined to signals of the simplest process class, namely the stationary gaussian processes. In numerical tests of digital terrain models (DTMs) rather good agreement with empirical compression rates has been obtained, but significant differences showed in special cases. To settle these cases requires an advanced modelling, at least to include the processes with stationary increments.

In order to separate certain effects, first it will be investigated how resistant present estimation formulas of compression rates are to deviations from the normal distribution and what quantitative effect non-zero means of the WCs have (Section 2). Then the mean value influence is investigated on the example of wavelets of finite order (Section 3). Comparisons between pre-estimated and empirical compression rates follow (Section 4). Motivation is provided by

Example 1: DTM Schneealpe, Austria. Empirical compression rates.

Table 1 shows relevant data of three test regions of the DTM Schneealpe (Institute of Photogrammetry and Remote Sensing, Vienna University of Technology) with different mean inclinations of the relief characterized by an estimation value for $\mu := E\,|\mathrm{grad}\,h|$. The thresholds S has been chosen so that the standard deviations of the errors due to data compression do not exceed those of the (inclination-dependent) sampling errors $\hat{m}_h$.

As expected the empirical compression rates $\hat{k}_e$ decrease with increasing $\hat{\mu}$ for, in general, the less (negligible) WCs fall into the interval $(-S, +S)$ the more the mean value $\bar{w}$ of the WCs is different from zero. This can be proofed for the individual cases (Sections 2 to 4). There is a particularly conspicuous jump from $\hat{k}_e^{(1)}$ for wavelets of the 1^{st} order to $\hat{k}_e^{(2)}$ for wavelets of the 2^{nd} order, whereas no distinct order-dependent increase of $\hat{k}_e$ is observed for wavelets from the 2^{nd} order (experiments were made with Daub 4, 6, 8 of 2^{nd}, 3^{rd}, 4^{th} order as well as with Symmlet 4 of 4^{th} order and Coiflet 3 of 6^{th} order). Looking at the last column of Table 1, signal inhomogeneities concerning the mean values, i. e. the mean relief inclination, should dominate. This, however, is to be proved in detail (Section 3).

2 Probability distributions

2.1 Transformation of the moments

Each distribution or density function is defined by their set of moments. The problem whether a unique composition exists is referred to as the STIELTJES'S moment problem. For "simple" distributions such a solution always exists. This opens up - at least in theory - a possibility to transfer distributions of signal values in the original domain to the scales in the WD: Calculate the moments of the input distribution, transform them one after the other into the WD and recompose them according to the moment rule. The transformation of the 1^{st} moment is the simplest (cf. expectation values in Section 3). For the 2^{nd} moment the input ACF C_{ff} has to be convoluted twice with the wavelet; the complete correlation properties of f enter the variances in the WD. For illustration see

Example 2: Scale-dependent variances of a periodic signal in the WD.

The ACF of

$$f(t) = A_0 + A \sin \omega_0 t \tag{2.1}$$

is

$$C_{ff}(\tau) = \sigma_f^2 \cos \omega_0 \tau, \quad \sigma_f^2 = \frac{A^2}{2}. \tag{2.2}$$

In the WD on each scale $a > 0$

$$\sigma_w^2(a) = 2\pi a \sigma_f^2 \frac{|\hat{\psi}(-a\omega_0)|^2}{c_\psi} \tag{2.3}$$

holds with

$$c_\psi := 2\pi \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, d\omega = \text{const}, \tag{2.4}$$

cf. also MEIER (2003, Table 1, p. 58). The frequency ω_0 that defines the correlation behaviour transfers itself to the variances (2.3) in the WD.

Generally, for the transformation of the n^{th} moments ($n \geq 2$) so-called multi-point correlation functions (MPCFs) are needed, which have to be convoluted n times with the wavelet. Obviously it is difficult to realise such a programme. Although the MCPFs created within the statistical theory of turbulence by GRAFAREND (1972) were structurally investigated for purposes of non-linear prediction, one has difficulty already in estimating the conventional ACF (2-PCF) (SUTOR, 1997). Fortunately, there are some distributions which allow one to reach one's goal with ease.

2.2 Normal distribution

If the values of the input signal f are normal-distributed, then also the WCs are normal-distributed due to linearity of the WT. If, moreover, f is stationary, then the mean values $\bar{w}(a)$ become zero and the WCs are distributed on each scale $a > 0$ as $N(0, \sigma_w^2(a))$ (cf. Section 3). To include the cases with $\bar{w} \neq 0$, compression components of WC are calculated from $N(\bar{w}(a), \sigma_w^2(a))$ as follows.

The relative number of WCs falling below a given threshold value S on each scale follows as

$$\begin{aligned}\lambda_N &= P(-S < w < +S) \\ &= P(g_1 < \frac{w-\bar{w}}{\sigma_w} < g_2) \\ &= F(g_2) - F(g_1) = \frac{1}{\sqrt{2\pi}} \int_{g_1}^{g_2} e^{-t^2/2} dt,\end{aligned} \tag{2.5}$$

with $F(g_{1,2})$ as the distribution function of $N(0,1)$ at the positions $g_1 = -(S+\bar{w})/\sigma_w$, $g_2 = (S-\bar{w})/\sigma_w$. Special cases of (2.5) are given in Table 2. The approximation formulas are generated by Taylor expanding $F(g_{1,2})$ breaking off after the quadratic term. For that let us look at

Example 3: Approximation for large mean values ($|\bar{w}| \gg S$)

$$\begin{aligned}F(g_1) &= F(-\frac{\bar{w}}{\sigma_w} - \frac{S}{\sigma_w}) \\ &\approx F(-\frac{\bar{w}}{\sigma_w}) - \frac{S}{\sigma_w} F'(-\frac{\bar{w}}{\sigma_w}) + \frac{1}{2}(\frac{S}{\sigma_w})^2 F''(-\frac{\bar{w}}{\sigma_w}) \ , \\ F(g_2) &= F(-\frac{\bar{w}}{\sigma_w} + \frac{S}{\sigma_w}) \\ &\approx F(-\frac{\bar{w}}{\sigma_w}) + \frac{S}{\sigma_w} F'(-\frac{\bar{w}}{\sigma_w}) + \frac{1}{2}(\frac{S}{\sigma_w})^2 F''(-\frac{\bar{w}}{\sigma_w}) \ , \\ \lambda_N &= F(g_2) - F(g_1) \approx 2\frac{S}{\sigma_w} F'(-\frac{\bar{w}}{\sigma_w}) \\ &= \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w} e^{-\bar{w}^2/2\sigma_w^2} < \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w} \ .\end{aligned} \tag{2.6}$$

That quantifies the influence of $\bar{w} \neq 0$ on λ_N. To check the resistance of such estimation formulas to deviations from the normal distribution a test distribution is recommended that deviates from the normal distribution to an extreme extent.

2.3 Distribution of the ordinates of a periodic signal

The density function of the ordinates of the signal (2.1) is (cf. e. g. MEIER and KELLER (1990), p. 43)

$$g(x) = \pi^{-1}[A^2 - (x - A_0)^2]^{-1/2}, \quad |x - A_0| < A\,. \tag{2.7}$$

It behaves totally different to that of the normal distribution (Fig. 1).

If a signal (2.1) is transformed by wavelets of n^{th} order, the n^{th} derivation is when passing the limit $a \to 0$ approximated as

$$f^{(n)}(t) = \omega_0^n A sin[\omega_0 t + n(\frac{\pi}{2})] \tag{2.8}$$

whereby some proportionality, or normalization factors, respectively, are to be taken into consideration; cf. e. g. BEYER and MEIER (2001). That means that in the WD the type of the distribution with the density (2.7) is approximately maintained at least on lower scales: In (2.7) the constant becomes $A_0 = 0$ and the amplitude $\omega_0^n A =: A_w$; the phase shift is of no importance. If in (2.1) instead of the constant there is a position-dependent trend, so there are cases with $\bar{w} \neq 0$ and in (2.7) A_0 is to be replaced with $\bar{w}$. On lower scales, where there is the proportionately strongest compression, therefore

$$\begin{aligned}\lambda_0 &= P(-S < w < +S) \\ &\approx \frac{1}{\pi} \int_{-S}^{+S} \frac{dx}{[A_w^2 - (x-\bar{w})^2]^{1/2}} \\ &= \frac{1}{\pi}\{\arcsin(\frac{S+\bar{w}}{A_w}) + \arcsin(\frac{S-\bar{w}}{A_w})\} \ ,\end{aligned} \tag{2.9}$$

Table 2: Special cases of compression components λ by thresholding, normally distributed (λ_N) and periodically oszillating wavelet transforms (λ_0). Discussion in Sections 2.2, 2.3.

Special cases	λ_N	Remarks	λ_0	Remarks
$\bar{w} = 0$	$2F\left(\frac{S}{\sigma_w}\right) - 1$	$F(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$	$\frac{2}{\pi} \arcsin\left(\frac{\sqrt{2}}{2} \frac{S}{\sigma_w}\right)$	$\sigma_w^2 = \frac{A_w^2}{2}$, $\frac{S}{\sigma_w} \le \sqrt{2}$
$\bar{w} = 0,\ S \ll \sigma_w$	$\approx 2\frac{S}{\sigma_w} F'(0) = \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w}$	$F'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$\approx \frac{\sqrt{2}}{\pi} \frac{S}{\sigma_w}$	$\frac{\lambda_N}{\lambda_0} \approx \sqrt{\pi}$
$\bar{w} \ne 0,\ \lvert\bar{w}\rvert \ll S$	$\approx 2F\left(\frac{S}{\sigma_w}\right) + \left(\frac{\bar{w}}{\sigma_w}\right)^2 F''\left(\frac{S}{\sigma_w}\right) - 1$	$F''(x) = -\frac{x}{\sqrt{2\pi}} e^{-x^2/2}$	$\approx \frac{1}{\pi}\Big\{2 \arcsin\left(\frac{\sqrt{2}}{2} \frac{S}{\sigma_w}\right)$	$\frac{S}{\sigma_w} \le \sqrt{2}$
	$= 2F\left(\frac{S}{\sigma_w}\right) - \frac{1}{\sqrt{2\pi}} \left(\frac{\bar{w}}{\sigma_w}\right)^2 \frac{S}{\sigma_w} e^{-S^2/2\sigma_w^2} - 1$		$+ \frac{\sqrt{2}}{4} \frac{S}{\sigma_w} \left(\frac{\bar{w}}{\sigma_w}\right)^2 \left[1 - \frac{1}{2}\left(\frac{S}{\sigma_w}\right)^2\right]^{-3/2}\Big\}$	$\frac{\lambda_N}{\lambda_0} < \sqrt{\pi}$
	$< 2F\left(\frac{S}{\sigma_w}\right) - 1$		$> \frac{2}{\pi} \arcsin\left(\frac{\sqrt{2}}{2} \frac{S}{\sigma_w}\right)$	
$\bar{w} \ne 0,\ \lvert\bar{w}\rvert \ll S \ll \sigma_w$	$\approx \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w}$		$\approx \frac{\sqrt{2}}{\pi} \frac{S}{\sigma_w}$	$\frac{\lambda_N}{\lambda_0} \approx \sqrt{\pi}$
$\bar{w} \ne 0,\ \lvert\bar{w}\rvert = S$	$F\left(2\frac{S}{\sigma_w}\right) - \frac{1}{2}$		$\approx \frac{1}{\pi} \arcsin\left(\sqrt{2}\frac{S}{\sigma_w}\right)$	$\frac{S}{\sigma_w} \le \frac{\sqrt{2}}{2}$
$\bar{w} \ne 0,\ \lvert\bar{w}\rvert \gg S$	$\approx 2\frac{S}{\sigma_w} F'\left(\frac{\bar{w}}{\sigma_w}\right) = \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w} e^{-\bar{w}^2/2\sigma_w^2}$		$\approx \frac{\sqrt{2}}{\pi} \frac{S}{\sigma_w} \left[1 - \frac{1}{2}\left(\frac{\bar{w}}{\sigma_w}\right)^2\right]^{-1/2}$	$\frac{\bar{w}}{\sigma_w} \le \sqrt{2}$
	$< \frac{2}{\sqrt{2\pi}} \frac{S}{\sigma_w}$		$> \frac{\sqrt{2}}{\pi} \frac{S}{\sigma_w}$	$\frac{\lambda_N}{\lambda_0} < \sqrt{\pi}$

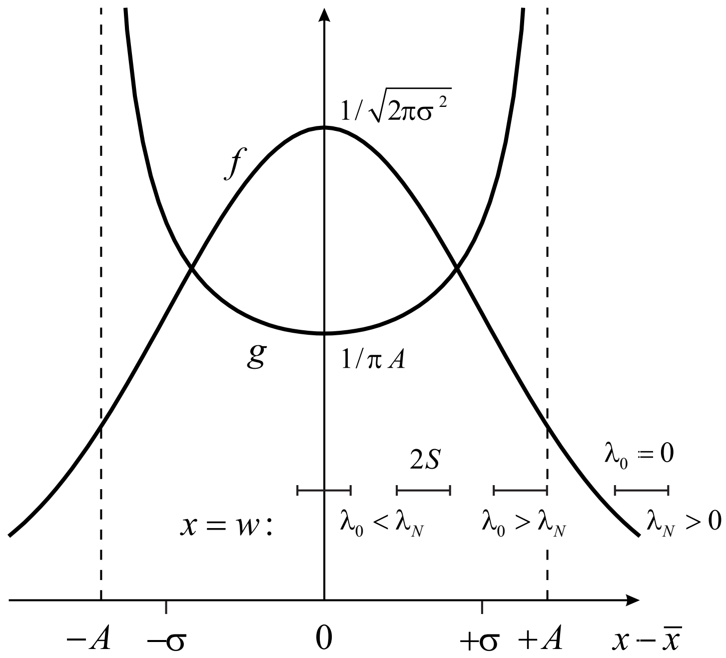

Fig. 1: Density functions of the normal distribution and of the distribution of the ordinates of a periodic sinal with equal first and second order moments. Position of the threshold intervals $2S$, cases $\bar{x} = \bar{w} = 0$, $\bar{x} = \bar{w} \neq 0$. Discussion of the resulting compression components λ_N, λ_0 see Section 2.3.

with $A_w = \sqrt{2}\sigma_w$. Special cases are indicated in Table 2; the approximation formulas were obtained as mentioned above. The differences to the normal distribution are seen in Fig. 1, where threshold intervals of the width $2S$ were entered: If $\bar{w} = 0$ or $|\bar{w}| \neq 0$, but not too big, then $\lambda_0 < \lambda_N$. If $|\bar{w}|$ continues to increase, then $\lambda_0 > \lambda_N$. If eventually $|\bar{w}| \geq A_w + S$, i. e. when the signals in the WD outside the interval $(-S, +S)$ oscillate, then λ_0 falls to zero, while λ_N takes there a finite, albeit small value. Now let λ_0 be quantitatively considered in comparison to λ_N.

Example 4: Behaviour of the compression components.

In Fig. 2 λ_N, λ_0 and the ratio λ_N/λ_0 are plotted for the cases $\bar{w} = 0$ and $\bar{w} = S$. For small $|\bar{w}|$ and small S, $1 < \lambda_N/\lambda_0 < \sqrt{\pi}$ applies. For big S and/or $|\bar{w}|$ this ratio can change to $\lambda_N/\lambda_0 < 1$. Let especially be $|\bar{w}| = A_w \gg S$. By a series expansion as above it is found:

$$\lambda_N \approx \frac{2/e}{\sqrt{2\pi}} \frac{S}{\sigma_w}, \quad \lambda_0 \approx \frac{\sqrt{2}}{\pi} \frac{S}{\sigma_w}, \quad \frac{\lambda_N}{\lambda_0} \approx \frac{\sqrt{\pi}}{e} \approx 0.65\,. \tag{2.10}$$

Although both distributions behave quite differently, the compression components on the scales for not too big S, $|\bar{w}|$ differ by less than the factor two. From this we can conclude that the estimation formula (2.5) and the special cases in Table 2 are resistant to (partly even rather big) deviations from the normal distribution and can particularly be used to wavelet-transformed height data of DTM without hesitation (Section 4). Further, in both distributions the strong influence of the mean values $\bar{w} = \bar{w}(a)$ on $\lambda = \lambda(a)$ can be seen. What, now, are the prerequisites for $\bar{w} = 0$ or $\bar{w} \neq 0$?

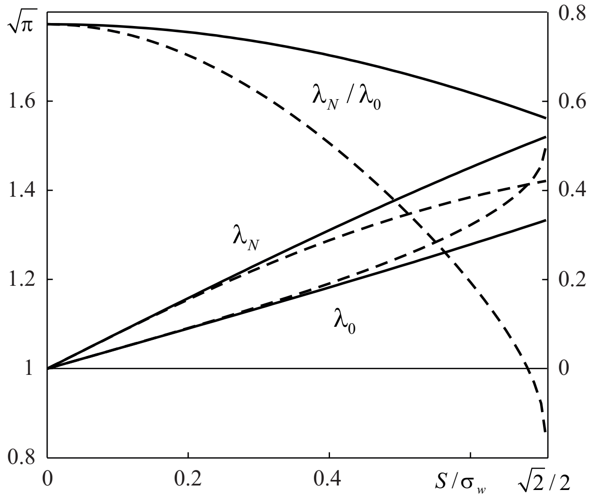

Fig. 2: Compression components of normally distributed (λ_N) and periodic oszillating wavelet transforms (λ_0), scaling right, relation λ_N/λ_0, scaling left, in two cases: $\bar{w} = 0$ (full lines), $|\bar{w}| = S$ (dashed lines), according to the results in Table 2.

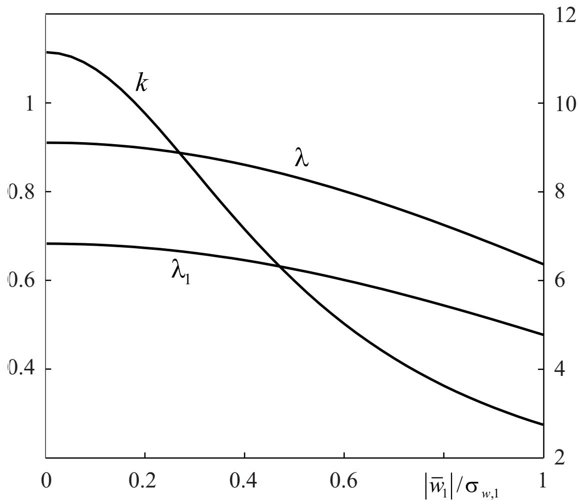

Fig. 3: Data compression with the Haar wavelet and threshold $S = \sigma_w(a_1)$. Compression component $\lambda_1 = \lambda(a_1)$, total proportion λ (scaling left), total compression rate k (scaling right). Discussion in Section 4, example 7.

3 Expectation values

Let $f(t)$ be the realization of a stochastic process $F(t)$. When all realizations $f_i \subset F$ are put into the integral of the WT and the expectation value is formed, then

$$E\{w(a,b)\} = \frac{1}{\sqrt{c_\psi |a|}} \int_{-\infty}^{+\infty} E\{F(t)\} \psi\left(\frac{t-b}{a}\right) dt, \tag{3.1}$$

because the formation of the expectation value and the integration, as linear operations, can be exchanged. In general, the WT provides scale-independent expectation value *functions*. There are, however, distinctive special cases where (3.1) no longer depends on the position b.

Example 5: Stationary processes.

Let F be stationary with $E\{F(t)\} =: \bar{F} = const$. In this case $\bar{F}$ can be written in front of the integral so that (with the substitution $(t-b)/a =: x$)

$$E\{w(a)\} =: \bar{w}(a) = \sqrt{\frac{|a|}{c_\psi}} \bar{F} \int_{-\infty}^{+\infty} \psi(x) dx \equiv 0 \tag{3.2}$$

becomes because of

$$\int_{-\infty}^{+\infty} \psi(x) dx = 0\,. \tag{3.3}$$

The expectation values are - irrespective of the wavelet used - equal to zero on all scales.

Example 6: Processes with stationary increments.

Processes are understood as processes with stationary increments, if their first derivation is stationary, for example, $F_1(t) = \alpha t + F(t)$, where a stationary process $F(t)$ like in example 5 is additively superimposed on a linear trend αt, $\alpha = \text{const}$. Showing consideration for (3.2), only the term

$$E\{w(a)\} = \frac{\alpha}{\sqrt{c_\psi |a|}} E\left\{\int_{-\infty}^{+\infty} t\psi\left(\frac{t-b}{a}\right) dt\right\}$$

remains, and with the substitution as above

$$E\{w(a)\} = \alpha\sqrt{\frac{|a|}{c_\psi}}E\left\{a\int_{-\infty}^{+\infty} x\psi(x)dx\ +b\int_{-\infty}^{+\infty} \psi(x)dx\right\}\,.$$

The second integral becomes zero because of (3.3), the first one corresponds to the 1^{st} moment μ_1 of the wavelet ψ, and only does not become zero if ψ is of 1^{st} order. For such wavelets constant values result on each scale,

$$\bar{w}(a) = \frac{\alpha\mu_1 a^{3/2}}{\sqrt{c_\psi}} \neq 0\,, \tag{3.4}$$

otherwise also here $\bar{w}(a) \equiv 0$.

Among all signals the mean value-stationary signals can - as the estimation functions in Table 2 indicate - be compressed most intensely. Further, with (3.2), (3.4), the jump in the empirical compression rates, example 1, Table 1, can be explained qualitatively: The used wavelets of 2^{nd} to 6^{th} order with $\bar{w}(a) = 0$ on all scales without exception compress distinctly more intense than 1^{st} order wavelets with $\bar{w}(a) \neq 0$. The influence of different (wavelet-dependent) $\sigma_w(a)$, however, is rather small. A quantitative estimation follows in Section 4, example 8.

4 Comparison of compression rates

Compression rates are defined in the literature in different ways, as a rule, for the WT as the ratio of the incompressed to the compressed data. Let N be the number of the original signal values or the WCs without compression, respectively, and λ the proportion of the WCs eliminated by the thresholding method, then the compression rate due to the mentioned measures is

$$k = \frac{N}{N - N\lambda} = \frac{1}{1-\lambda}, \quad \lambda = \sum_{k=1}^{n} 2^{-k}\lambda(a_k)\,, \tag{4.1}$$

cf. Meier (2003, p. 33). The author succeeded in creating a closed presentation of the sum of the proportions λ_k over all scales a_k ($k = 1, 2, \ldots, n$) only for the Haar wavelet.

Example 7: Compression with the Haar wavelet.

Provided the ACF C_{ff} of the input signal is of the bell-shaped curve type,

$$\frac{4}{3} < \frac{\lambda}{\lambda_1} < 2 \quad \text{for} \quad 1 > \rho_1 > 0\,, \tag{4.2}$$

follows (Meier 2003, p. 37), with ρ_1 as the correlation coefficient of directly neighbored signal values. In actual DTMs, as a rule, $0.9 < \rho_1 < 1$ applies (Borkowski 1994). In Fig. 3 the boundary case

$$\lambda = \frac{4}{3}\lambda_1 \quad \text{for} \quad \rho_1 \to 1\,, \tag{4.3}$$

and the resulting minimum rate according to (4.1) are shown. With the special threshold $S = \sigma_w(a_1)$ it follows from (2.5):

$$\lambda_1 = F\left(1 - \frac{\bar{w}_1}{\sigma_{w,1}}\right) + F\left(1 + \frac{\bar{w}_1}{\sigma_{w,1}}\right) - 1\,. \tag{4.4}$$

The quick drop of the compression rate with growing $|\bar{w}_1| = |\bar{w}(a_1)|$ is seen.

Let's conclude by returning to the introducing example 1 and compare the compression rates pre-estimated for the one-dimensional case using formula (2.5) to empirically determined compression rates

Table 3: DTM Schneealpe, Austria. A priori and empirically estimated compression components and compression rates. Discussion in Section 4, example 8.

		Wavelet of 1^{st} order (Haar)						Wavelet of 2^{st} order (Daub4)				
Test	$S = \hat{m}_h$	$\lvert\hat{\bar{w}}_1\rvert$	$\hat{\sigma}_{w,1}$	$\hat{\lambda}_1$	$\hat{k}_1$	$\hat{k}$	$\hat{k}_e$	$\lvert\hat{\bar{w}}_1\rvert$	$\hat{\sigma}_{w,1}$	$\hat{\lambda}_1$	$\hat{k}_1$	$\hat{k}_e$
1	2.2	1.50	4.50	0.354	1.55	1.9	1.9	0.067	2.73	0.580	2.38	2.8
		0.405	4.08	0.408	1.69	2.2		0.067	2.67	0.588	2.43	
2	2.8	0.857	6.26	0.342	1.52	1.8	1.5	0.051	3.18	0.622	2.65	2.9
		0.192	7.71	0.284	1.40	1.6		0.011	2.94	0.660	2.93	
3	3.1	0.286	9.80	0.248	1.33	1.5	1.4	0.089	3.99	0.563	2.28	2.8
		3.05	7.13	0.308	1.45	1.7		0.045	3.17	0.560	2.27	

for the two-dimensional case. Such a comparison is admissible, if the compression was performed using tensor-product wavelets, i.e. signal decomposition relative to axes and diagonals of the discrete grid (Beyer 2002).

Example 8: DTM Schneealpe, Austria. A-priori and empirically estimated compression components and compression rates.

Table 3 contains selected data concerning two wavelets. The threshold value, equal to the pre-estimated height error, was set lower than in example 1 out of consideration for the data quality. The estimated values of $|\bar{w}_1|$, $\sigma_{w,1}$ relative to the axes indicate that the 2D signal is neither homogeneous nor isotropic. Also the deviations from the normal distribution (not represented) are significant. Nevertheless, the $\hat{k}$ pre-estimated for the Haar wavelet from the (robust) formulas (2.5), (4.2), (4.3) correspond very well with the empirical values $\hat{k}_e$. For Daub4 no direct comparison is possible. It is seen, however, that - above all, because of the very small mean values compared to the Haar wavelet - the compression components $\hat{\lambda}_1$ and $\hat{k}_1$ turn out distinctly bigger and the compression of DTMs with lower threshold values should be useful only with 2^{nd} or higher order wavelets.

5 Conclusions

It is known that the possibilities to calculate using the continuous WT are limited, which has not been detrimental to the wide use of the discrete WT. This also applies to the statistical characteristics of the input signals: Model formations are only possible with a few (standard) wavelets and sufficiently simple process classes from which the signals originate. The author has been anxious to exploit these possibilities aiming at supporting the so far rather empirical investigations on data compression from a statistical point of view.

As a rule, the distribution of the signal values and the WCs is not known. A-priori estimations of the compression rates on the assumption of normal-distributed WCs are admissible and realistic, because they are resistant to (even large) deviations from the normal distribution. Among all signals those the WCs of which are, in the mean, equal to zero on all scales can be compressed most intensely. These include the mean value-stationary signals, then those having a mean value with a linear trend, if they are compressed using 2^{nd} or higher order wavelets. Variances have, compared to non-zero mean values in the WD, a smaller influence. Apart from the threshold values the compression rates depend - expressed in a simplified manner and as confirmed by empirical investigations - more on special signal properties than on the wavelets used.

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Prof. Dr.-Ing. habil. S. Meier,
Technische Universität Dresden
Institut für Planetare Geodäsie
01062 Dresden
Tel./Fax: 0351-463-33416/37063
E-Mail: meier@ipg.geo.tu-dresden.de

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