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THEORY OF STABILITY OF LAYERED CYLINDRICAL RODS IN ELASTO-PLASTIC STATES EXEMPLIFIED BY STEEL R 35

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ABSTRACT

The study presents theory of layered axially compressed cylindrical rods (shells with a core). The author analysed problems of critical loads in elasto-plastic states. The plasticity ratio of a critical transverse section was defined by the angle α , and was related to the slenderness ratio of rods. The obtained theoretical results were compared with the experimental results of rods made of R35 steel (Standard PN-73/H-74240).

Key words: Stability, Critical Force, Layered structure, Sandwich, Rod.

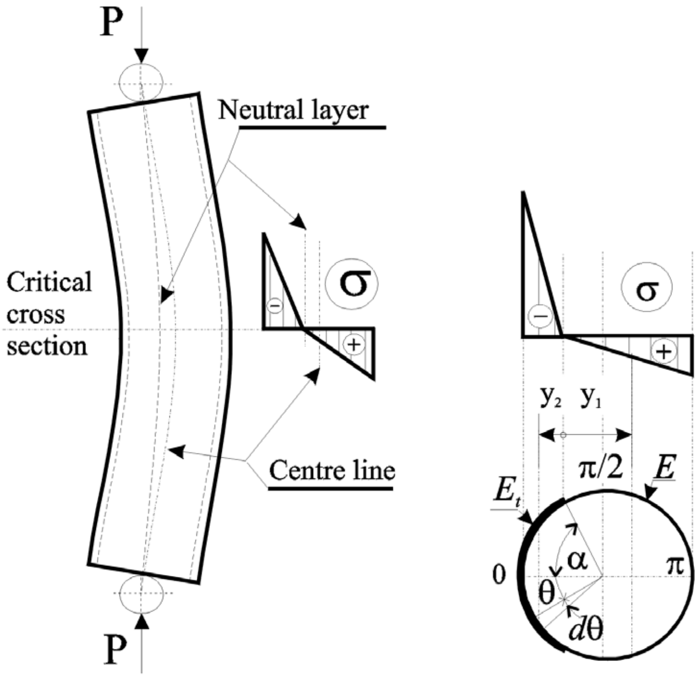
INTRODUCTION

Basic theory of stability of axially compressed slender rods in elasto-plastic states was formulated by Engesser (1889; 1895), Kármán (1908; 1910) and Shanley (1947). The theory presented in this paper was first described by Murawski (2002). According to this theory, in the case of a cylindrical rod (Fig.1), the distances y_1 and y_2 from the neutral layer to the segment dA in the critical cross-section (without taking ovality into consideration), and within the elastic or plastic zone, respectively, can be expressed at the moment of losing the stability by the following relationships:

$$dA = t\overline{R}\ \mathrm{d}\theta, \qquad y_1 = \overline{R}\ (\cos\alpha - \cos\theta), \qquad y_2 = \overline{R}(\cos\theta - \cos\alpha) \tag{1}$$

where $\overline{R}$ is the average radius of the tube, t – wall thickness, $2*\alpha$ -angle describing the plastic part of the critical cross-section, A – area of the critical cross-section.

Fig. 1. Stresses in critical cross-section axially compressed column according to the Engesser-Kármán-Shanley hypothesis



The equilibrium of forces due to the stresses in relation to the neutral layer is described with formulas

$$P + \int_{A_2} \sigma_2 dA_2 = \int_{A_1} \sigma_1 dA_1 \ , \qquad \sigma_2 = \frac{y_2}{\rho} E_t \quad , \qquad \sigma_1 = \frac{y_1}{\rho} E \quad , \tag{2}$$

where P denotes the force, A_1 , A_2 – area of the elastic and plastic part of the critical transverse cross section, respectively, ρ - curvature radius of the neutral layer, E – Young's modulus, E_t – tangent modulus. The equilibrium of moments in relation to the neutral layer is described with the formula:

$$\int_{A_2} \sigma_t y_2 dA_2 + \int_{A_1} \sigma_1 y_1 dA_1 = -Py \quad , \tag{3}$$

where y is the distance from the outside line of the load to the neutral layer. After integrating Eg. (3) and taking into consideration the axial moments of inertia J_1, J_2 of the tensioned and plastic part, respectively, in the critical cross-section, yields:

$$\frac{E_t J_2 + E J_1}{\rho} = -Py \quad . \tag{4}$$

Analysis of the differential equation of the deflection line of the neutral layer (on the assumption of small deformations of the line: $\frac{d^2 y}{dx^2} \cong \frac{1}{\rho}$ and of the existence of an equivalent modulus E_{EK}) yields the following equation of the deflection line of the neutral layer:

$$E_{EK} \; J \frac{d^2 y}{dx^2} + Py = 0 \quad \Rightarrow \quad \frac{E_{EK} J}{\rho} = -Py \quad . \tag{5}$$

Having integrated differential Eq. (5), we get the formula for the critical stress:

$$\sigma_{EK} = \left(\frac{\pi}{\lambda}\right)^2 E_{EK} = \left(\frac{\pi}{\lambda}\right)^2 \frac{E_t J_2 + EJ_1}{J} \quad . \tag{6}$$

where λ denotes the slenderness ratio (depending on boundary conditions). Taking into account Eq. (4), we determine a formula for the value of the equivalent modulus in relation to Young's modulus as:

$$\frac{E_{EK}}{E} = \left(\frac{1}{\pi} + \frac{1}{\operatorname{tg}\alpha - \alpha}\right)\left[\alpha\left(1 + \frac{\cos 2\alpha}{2}\right) - \frac{3\sin 2\alpha}{4}\right] + 1 - \frac{\alpha}{\pi} + \frac{1-\alpha}{2}\cos 2\alpha + \frac{3\sin 2\alpha}{4\pi} \quad . \tag{7}$$

However, this function has a physical sense only for $\alpha \in$ 90°, 138°), because the following should hold:

$$1 \geq E_{EK}/E > 0.$$

From Eqs. (1), (2):

$$\frac{E_t}{E} = \frac{\sin\alpha + (\pi - \alpha)\cos\alpha}{\sin\alpha - \alpha\cos\alpha} = 1 + \frac{\pi}{\operatorname{tg}\alpha - \alpha} \quad . \tag{8}$$

The above function has a physical meaning only for $\alpha \in$ (90°, 180°), because the condition: $1 \geq E_t/E > 0$ should be satisfied.

Due to the above-mentioned limitations, the author made his own analysis of the stability of thin-walled rods. He assumed that in an elastic state the loss of carrying capacity follows the exit of the resultant neutral layer from the critical transverse section, whereas in an elasto-plastic state - after the entry of the resultant neutral layer into the plastic zone. Therefore, the author assumed that the state of stresses in the critical transverse section after the loss of stability and before the loss of carrying capacity results from the superposition of pure compression and bending (Fig.2). Hence, see Murawski (1992):

$$\Sigma P = P_{kr} - P_m - P_H = 0 \Rightarrow P_{kr} = P_m + P_H \quad , \tag{9}$$

$$\Sigma M = P_{kr}\, y - P_m\, y_m + P_H\, y_H = 0 \quad , \tag{10}$$

where:

$$P_H = \sigma_H A_H = \sigma_H (\pi - \alpha)\overline{R}\cdot t \,, \quad P_m = \sigma_{kr}(\lambda \cong 0) A_m = \sigma_{kr}(\lambda \cong 0)\cdot \alpha \cdot \overline{R}\cdot t \,, \tag{11}$$

σ_H is the elasticity limit at compression (variable for different λ and corresponding α) and A_H , A_m denote the areas of the transverse section in the elastic and plastic state, respectively. Hence:

$$\sigma_{kr} = \frac{P_{kr}}{A} = \sigma_H \frac{2(\pi - \alpha)\overline{R}t}{2\pi\overline{R}t} + \sigma_{kr}(\lambda_1)\frac{2\alpha\overline{R}t}{2\pi\overline{R}t} = \sigma_H\left(1 - \frac{\alpha}{\pi}\right) + \sigma_{kr}(\lambda_1)\frac{\alpha}{\pi} \quad , \tag{12}$$

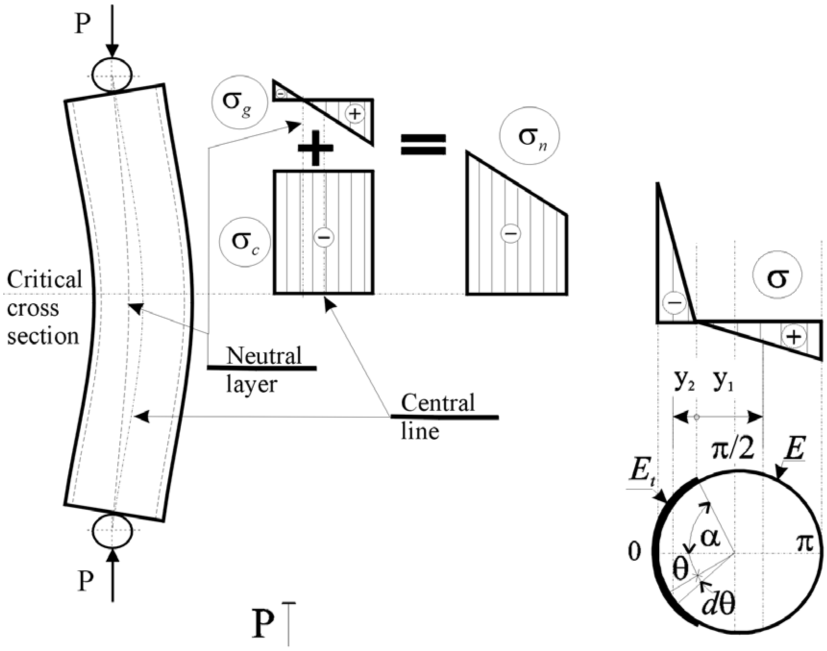
If P_H is attained for λ_{gr} then $P_{kr}(\lambda)$ from the range $(\lambda \cong 0; \lambda_{gr})$ is attained for the slenderness ratio:

$$\lambda = \lambda_{gr} - \frac{\alpha}{\pi}\lambda_{gr} \quad \Rightarrow \quad \frac{\alpha}{\pi} = \frac{\lambda_{gr} - \lambda}{\lambda_{gr}} \ , \tag{13}$$

From the above, we get:

$$\sigma_{kr}(\lambda) = \sigma_H(\lambda) + \frac{\lambda_{gr} - \lambda}{\lambda_{gr}}\left[\sigma_{kr}(0) - \sigma_H(\lambda)\right] \ . \tag{14}$$

Fig. 2. Stresses, in critical transverse section of axially compressed tube, after losing the stability according to the author's hypothesis



The stress $\sigma_{kr}\ (\lambda \cong 0)$ has beenacknowledged as the characteristic parameter for a given material and marked by R_e*. It follows from the investigation that the function $\sigma_H(\lambda)$ is of the first order, so the obtained function $\sigma_{kr}(\lambda)$ is of the second order. The stress $\sigma_H\ (\lambda \cong 0)$ has been acknowledged as the characteristic parameter as well and was marked by R^*_H. The linear function $\sigma_H(\lambda)$, has been described with formulas:

$$\sigma_H(\lambda) = R_H^{Eu} + \left(1 - \frac{\lambda}{\lambda_{gr}}\right)\left[R_H^* - R_H^{Eu}\right] \ , \qquad R_H^{Eu} = \sigma_H(\lambda_{gr}) = \sigma_{kr}(\lambda_{gr}) \ , \tag{15}$$

where R_H^{Eu} is the elasticity limit, used in the Euler formula to determine λ_{gr}. Then:

$$\sigma_{kr}(\lambda) = \left(1 - \frac{\lambda}{\lambda_{gr}}\right)\left(R_e^* + R_H^* \frac{\lambda}{\lambda_{gr}}\right) + R_H^{Eu}\left(\frac{\lambda}{\lambda_{gr}}\right)^2 \ . \tag{16}$$

After insertion of the Euler formula one gets:

$$\sigma_H(\lambda) = \left(\frac{\pi}{\lambda_{gr}}\right)^2 E + \left(1 - \frac{\lambda}{\lambda_{gr}}\right)\left[R_H^* - \left(\frac{\pi}{\lambda_{gr}}\right)^2 E\right], \ \sigma_{kr}(\lambda) = \left(1 - \frac{\lambda}{\lambda_{gr}}\right)\left(R_e^* + R_H^* \frac{\lambda}{\lambda_{gr}}\right) + E\left(\frac{\pi\lambda}{\lambda_{gr}^2}\right)^2, \tag{17}$$

or:

$$\sigma_H(\lambda)=R_H^{Eu}+\left(1-\frac{\lambda}{\pi}\sqrt{\frac{R_H^{Eu}}{E}}\right)\left[R_H^{*}-R_H^{Eu}\right],\sigma_{kr}(\lambda)=\left(1-\frac{\lambda}{\pi}\sqrt{\frac{R_H^{Eu}}{E}}\right)\left(R_e^{*}+R_H^{*}\frac{\lambda}{\pi}\sqrt{\frac{R_H^{Eu}}{E}}\right)+\frac{1}{E}\left(\frac{\lambda}{\pi}R_H^{Eu}\right)^2 . \quad (18)$$

The above formulas are suitable to use if the following parameters: $R_e^*, R_H^*, E, R_H^{Eu}$ or λ_{gr} are known. The first two should be determined from a compression test of stocky rods with λ close to 0 .

STABILITY OF LAYERED RODS

The influence of the core can be taken into account by adding its stiffness to the stiffness of the tube, because it acts parallelly with the tube in the direction of the load. The stiffness and elasticity, according to Hooke's formula,

$$P=\left(\frac{A}{a}E\right)\Delta a \quad , \qquad P=k\Delta a , \qquad (19)$$

are described with expressions:

$$k_{shell}=\frac{A_{shell}}{a}E \quad , \qquad k_{core}=\frac{A_{core}}{a}E_{core} \quad , \qquad E_{zr}=E+\frac{A_{core}}{A_{shell}}E_{core} \quad . \qquad (20)$$

where a denotes the initial length, Δa – the shortening, $k_{(shell,\ core)}$ – stiffness of the tube or core, $A_{(shell,\ core,)}$ - area of the cross section of the tube or core, $E_{(zr,\ PUR)}$ - coefficient of the longitudinal elasticity (of reference and of the core, respectively). Taking into account that the core made e.g. of polyurethane foam attains its own compression limit R_c at α much less shortening than the tube reaches its own yield limit and the reference limits:

$$R_{e\_zr}^{*}=R_e^{*}+\frac{A_{core}}{A_{shell}}R_c \quad , \qquad R_{H\_zr}^{*}=R_H^{*}+\frac{A_{core}}{A_{shell}}R_c \quad , \qquad R_{H\_zr}^{Eu}=R_H^{Eu}+\frac{A_{core}}{A_{shell}}R_c \quad , \qquad (21)$$

it enables one, with the help of formulas (15) and (16), to determine the stresses in the layered rods:

$$\sigma_H^{layered}(\lambda)=\left(\frac{\pi}{\lambda_{gr}}\right)^2 E_{zr}+\left(1-\frac{\lambda}{\lambda_{gr}}\right)\left[R_H^{*}-\left(\frac{\pi}{\lambda_{gr}}\right)^2 E_{zr}\right] \quad , \qquad (22)$$

$$\sigma_{kr}^{layered}(\lambda)=\left(1-\frac{\lambda}{\lambda_{gr}}\right)\left(R_{e\_zr}^{*}+R_{H\_zr}^{*}\frac{\lambda}{\lambda_{gr}}\right)+E_{zr}\left(\frac{\pi\lambda}{\lambda_{gr}^2}\right)^2 \quad , \qquad (23)$$

or

$$\sigma_H^{layered}(\lambda)=R_{H\_zr}^{Eu}+\left(1-\frac{\lambda}{\pi}\sqrt{\frac{R_{H\_zr}^{Eu}}{E_{zr}}}\right)\left[R_H^{*}-R_H^{Eu}\right] \quad , \qquad (24)$$

$$\sigma_{kr}^{layered}(\lambda)=\left(1-\frac{\lambda}{\pi}\sqrt{\frac{R_{H\_zr}^{Eu}}{E_{zr}}}\right)\left(R_{e\_zr}^{*}+R_{H\_zr}^{*}\frac{\lambda}{\pi}\sqrt{\frac{R_{H\_zr}^{Eu}}{E_{zr}}}\right)+\frac{1}{E_{zr}}\left(\frac{\lambda}{\pi}R_{H\_zr}^{Eu}\right)^2 \qquad (25)$$

Comparisons of graphs of the theoretical and approximated functions from the author's investigations, see Murawski (1998) ($R_e^* = 346{,}5$ MPa, $R_H^* = 268{,}24$ MPa, $E = 211000$ MPa, $\lambda_{gr} = 102{,}6$), are presented in Figs 3 and 4. The differences of values assumed by the stress functions in relation to the slenderness ratio do not exceed 6.9 % for the thin-walled rods and 7.7 % for the layered ones.

Fig. 3. Comparison of graphs of theoretical and approximated functions from researches for thin-walled rods

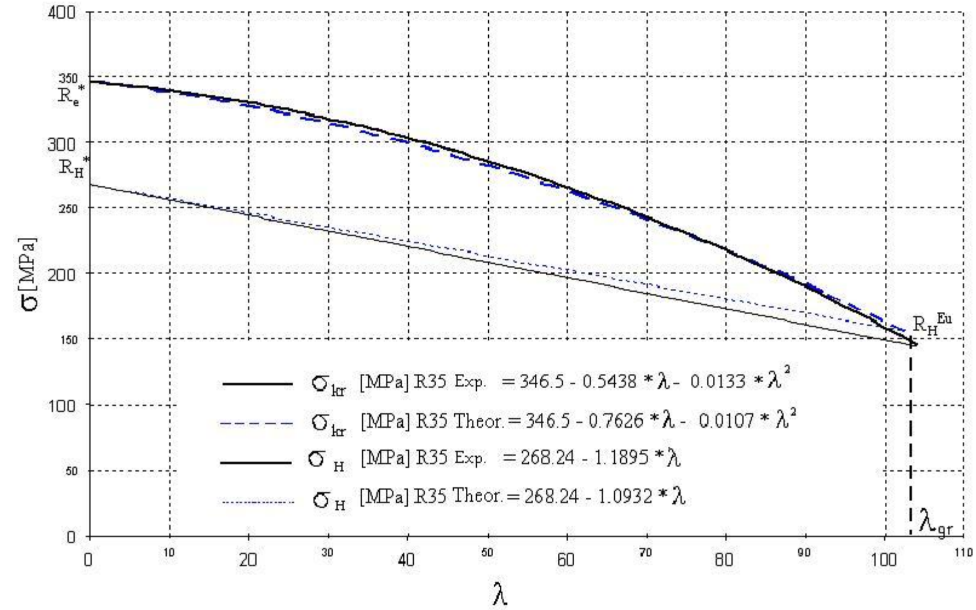
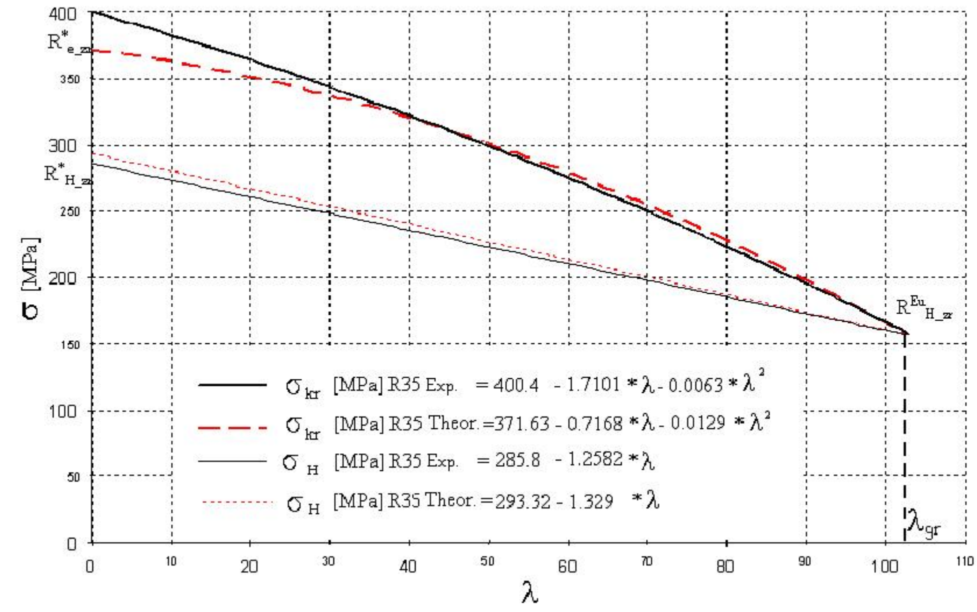


Fig. 4. Comparison of graphs of theoretical and approximated functions from the study of layered rods



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