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DIFFERENT METHODS OF MODELING VIBRATIONS OF PLATES MADE OF FUNCTIONALLY GRADED MATERIALS

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ABSTRACT

The subject of this paper are thin plates with a characteristic two-component structure: periodic in one direction and functionally variable in the second. This provides a gradient microstructure of the material properties of plates in the macro scale. The work is analyzed self-vibrations of this type of composites.

Modeling was made using three independent methods. The results obtained are summarized and compared in terms of accuracy and effort calculations. For comparison and evaluation of modeling methods were selected as follows:

- tolerant approximation technique proposed and described in the monograph [5], which was applied to this kind of plates for mechanical problems in [2][3][4] and for heat conduction problems in [1],
- finite element method, implemented with commercial computer program,
- heuristic method, which consisted of a simple, consistent with engineering intuition, averaged mechanical properties of the plate in one direction and bringing the problem to the problem of vibration of one-dimensional beams with variable cross-section.

The general results are illustrated by the analysis of a specific problem with different kind of boundary conditions plate band.

Key words: Self-vibrations, plate band, functionally graded materials, tolerant averaging technique.

1. INTRODUCTION

The subject of this paper are thin plates with a characteristic two-component structure: λ-periodic in one direction (along ξ^1 coordinate) and functionally variable in the second (along ξ^2 coordinate) (Fig. 1a). This provides a gradient microstructure of the material properties of plates in the macro scale. (Fig. 1b)

The main objective of the research is to apply a three independent models describing the dynamic behaviour of micro heterogeneous plate band made of two components. The general assumption of the research is that the generalized period λ is sufficiently small when compared to width of plate band L (Fig.1.a). The main attention is given to compare obtained results in terms of accuracy and effort calculations.

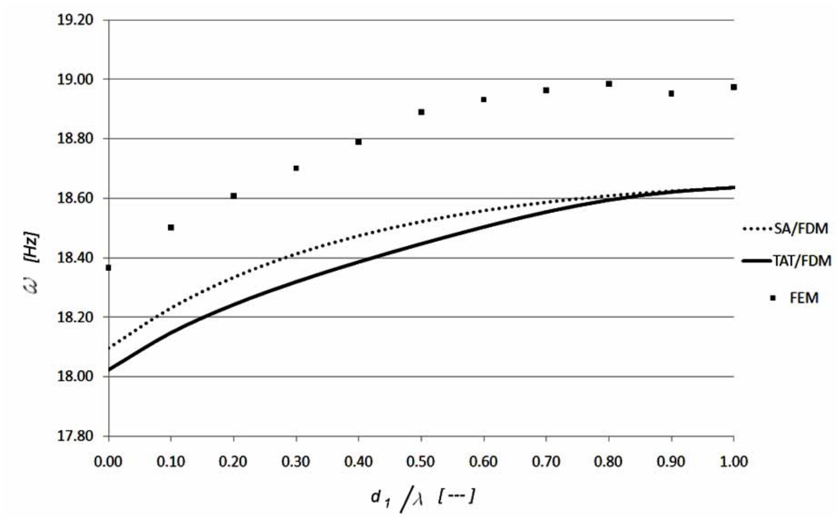

Fig. 1. Fragment of plates midplane with functionally graded microstructure: a) microscopic level, b) macroscopic level

Problems of plates of this kind were investigates by means of different methods. However, exact analysis of those plates within solid mechanics is too complicated to constitute the basis for solving most of the engineering problems. Thus, many different approximate modelling methods for functionally graded material plates were formulated.

In this paper we select three independent modeling methods as follows:

- tolerant approximation technique proposed and described in the monograph [5], which was applied to this kind of plates in [2][3][4]
- finite element method, implemented with commercial computer program, Autodesk Robot Structural Analysis 2011,
- heuristic method, which consisted of a simple, consistent with engineering intuition, averaged mechanical properties of the plate in one direction and bringing the problem to the problem of vibration of one-dimensional beams with variable cross-section.

2. MODELING

The base of modeling procedure are:

- strain-displacements relations:

$$\kappa_{\alpha\beta} = -w_{|\alpha\beta}, \tag{1}$$

where: $\kappa_{\alpha\beta}$ is curvature, w is displacement field.

- constitutive equations:

$$m^{\alpha\beta} = DH^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta}, \tag{2}$$

where:

$$H^{\alpha\beta\gamma\delta} = \frac{1}{2}\left\{ \begin{matrix} g^{\alpha\mu}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\mu} - \\ + \upsilon\left(\in^{\alpha\gamma}\in^{\beta\mu} + \in^{\alpha\mu}\in^{\beta\gamma}\right) \end{matrix} \right\}, \tag{3}$$

$$D = \frac{E\delta^3}{12\left(1-\upsilon^2\right)}, \tag{4}$$

E – Young module, δ – thickness of plate, ν – Poisson number, $\in^{ij}$ – component of Ricci tensor, g – component of contravariant metric tensor.

This equations have highly oscillating coefficients, so they are difficult to solve.

Next we use tolerance averaging technique (TAT) for modeling of dynamic behavior of thin plates. It was presenting in [5]. You can find many examples from TAT and bibliography in [4]. This theory defined some operators and lemates. The most important are:

- averaging operator:

$$\langle f\rangle(\xi^{\alpha})=\frac{1}{\lambda}\int_{\xi^1-\frac{\lambda}{2}}^{\xi^1+\frac{\lambda}{2}} f\left(y,\xi^2\right)dy\,, \tag{5}$$

where: y is local coordinates

- slowly varying function:

$$\begin{gathered} F(\cdot)\in SV_{\varepsilon}(T) \\ \Downarrow \\ \forall_{x,y\in\Pi}\, x-y\in\Delta \Rightarrow | DF(x)-DF(Y)|<\varepsilon_{DF} \end{gathered}\,, \tag{6}$$

where: ε_{DF} is tolerance parameter, Δ is periodicity cell, Π is the plate midplane,

$$DF\in\{F,\nabla F,...\}, \tag{7}$$

- Δ-periodic function

$$\begin{gathered} f\in L^{\infty}_{per}(\Delta) \\ \Downarrow \\ \exists_{x,x+L\in\Pi}\, f(x,\cdot,t)=f(x+L,\cdot,t) \end{gathered}\,, \tag{8}$$

- displacements field disjoined:

$$w(\cdot,t)=w^0(\cdot,t)+q^A(\cdot)V_A(\cdot,t), \tag{9}$$

where:

$$w^0(\cdot,t)\in SV_{\varepsilon}(T),\; V_A(\cdot,t)\in SV_{\varepsilon}(T),$$

are basic unknown, $q^A(\cdot)$ are known shape function.

- the most important theorem:

$$\langle fF\rangle(x)\cong\langle f\rangle F(x), \tag{10}$$

$$\langle f\nabla(hF)\rangle(x)\cong\langle fF\nabla h\rangle(x), \tag{11}$$

$$\langle f\nabla\nabla(hF)\rangle(x)\cong\langle fF\nabla\nabla h\rangle(x), \tag{12}$$

We receive first equation when we average equation of motion:

$$\left\langle m^{\alpha\beta}{}_{|\alpha\beta}+p-\rho\ddot{w}\right\rangle=0\,, \tag{13}$$

Next we multiply equation of motion by test functions and after averaging procedure we get remaining N equations:

$$\left\langle q^A\left(m^{\alpha\beta}{}_{|\alpha\beta}+p-\rho\ddot{w}\right)\right\rangle=0\,, \tag{14}$$

After substitution: constitutive equations, strain-displacements relations, displacement field disjoined, we get averaging equations:

$$\begin{gathered} \left\langle BH^{\alpha\beta\gamma\delta}w^0_{|\gamma\delta}\right\rangle_{\}\alpha\beta}+\left(\left\langle BH^{\alpha\beta11}q^A_{|11}\right\rangle V_A\right)_{|\alpha\beta}+\left(\left\langle BH^{\alpha\beta22}q^A\right\rangle V_{A|22}\right)_{|\alpha\beta}+\langle\rho\ddot{w}\rangle=\langle p\rangle\,, \\ \left\langle q^A{}_{|11}BH^{11\gamma\delta}\right\rangle w^0_{|\gamma\delta}+\left\langle q^A_{|11}BH^{1111}q^B_{|11}\right\rangle V_B+\left\langle q^A_{|11}BH^{1122}q^B\right\rangle V_{B|22}+\left(\left\langle q^A{}_{|11}BH^{22\gamma\delta}\right\rangle w^0_{|\gamma\delta}\right)_{|22}+ \\ +\left(\left\langle q^A BH^{2211}q^B_{|11}\right\rangle V_B\right)_{|22}+\left(\left\langle q^A BH^{2222}q^B\right\rangle V_{B|22}\right)_{|22}+\left\langle q^A\rho q^B\right\rangle\ddot{V}_B=\left\langle q^A p\right\rangle \end{gathered} \tag{15}$$

This is system $N+1$ differential equations. Coefficients in above system are continuous and slowly-varying functions.

Previous discussion was for any curvilinear coordinate system. To solve above equations for the specific case we introduce a rectangular Cartesian coordinate system. Next we consider example: free vibrations of thin plate band. This plate is shown on Figure 1 a. in rectangular coordinates. We make some assumptions:

- displacement field disjoined:

$$w = w^0 + qV\,, \tag{16}$$

where:

$$w^0 \in SV_\varepsilon(T),\; V \in SV_\varepsilon(T),$$

- no external loading: $p=0$
- harmonic vibration:

$$\begin{aligned} w^0(\xi^\alpha, t) &= \overline{w}(\xi^\alpha)\cos(\omega t),\\ V(\xi^\alpha, t) &= \overline{V}(\xi^\alpha)\cos(\omega t), \end{aligned} \tag{17}$$

- shape function:

$$q(\cdot) = \lambda^2\left(\cos\left(\frac{2\pi\xi^1}{\lambda}\right) + C\right), \tag{18}$$

where constant C we receive from equation:

$$\langle q\rho \rangle = 0\,, \tag{19}$$

as:

$$C = \frac{\lambda\xi^2\left(-\rho_1 + \rho_1\cos\left(\frac{2\pi d}{\xi^2\lambda}\right) - \rho_2 - \rho_2\cos\left(\frac{2\pi d}{\xi^2\lambda}\right)\right)}{2\pi\left(\rho_1 d + \rho_2\lambda\xi^2 - \rho_2 d\right)}\,, \tag{20}$$

After many mathematic transformations we receive 4-th row partial differential equations system which 2 unknown and 2 equations as follows:

$$\begin{aligned} &\left\langle BH^{2222} w^0_{,22} \right\rangle_{,22} + \left(\left\langle BH^{2211} q^A_{,11}\right\rangle V_A\right)_{,22} + \left(\left\langle BH^{2222} q^A\right\rangle V_{A,22}\right)_{,22} + \langle \rho\ddot{w}\rangle = \langle p \rangle\,,\\ &\left\langle q^A{}_{,11} BH^{1122}\right\rangle w^0_{,22} + \left\langle q^A_{,11} BH^{1111} q^B_{,11}\right\rangle V_B + \left\langle q^A_{,11} BH^{1122} q^B\right\rangle V_{B,22} + \left(\left\langle q^A{}_{,11} BH^{2222}\right\rangle w^0_{,22}\right)_{|22} +\\ &+ \left(\left\langle q^A BH^{2211} q^B_{,11}\right\rangle V_B\right)_{|22} + \left(\left\langle q^A BH^{2222} q^B\right\rangle V_{B,22}\right)_{,22} + \left\langle q^A \rho q^B\right\rangle \ddot{V}_B = \left\langle q^A p\right\rangle \end{aligned} \tag{21}$$

Coefficients in this equations system are smooth and continuous, and they can be find thanks to programs for symbolic calculations.

We must consider boundary conditions. We write boundary conditions for bracket (left side clamped, right side free) below:

$$w\,|_{\xi^2=0} = 0 \quad \text{and} \quad \left(\frac{\partial w}{\partial \xi^2}\right)|_{\xi^2=0} = 0\,, \tag{22}$$

$$\left(\frac{\partial^3 w}{\left(\partial \xi^2\right)^3}\right)|_{\xi^2=L} = 0 \quad \text{and} \quad \left(\frac{\partial^2 w}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=L} = 0\,, \tag{23}$$

because (16):

$$w^0\,|_{\xi^2=0_i}=0\,,\ \left(\frac{\partial w^0}{\partial \xi^2}\right)|_{\xi^2=0_i}=0\,,$$

$$V\,|_{\xi^2=0_i}=0\quad \left(\frac{\partial V}{\partial \xi^2}\right)|_{\xi^2=0_i}=0\,, \tag{24}$$

$$\left(\frac{\partial^3 w^0}{\left(\partial \xi^2\right)^3}\right)|_{\xi^2=L}=0\,,\ \left(\frac{\partial^2 w^0}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=L}=0\,,$$

$$\left(\frac{\partial^3 V}{\left(\partial \xi^2\right)^3}\right)|_{\xi^2=L}=0\quad \left(\frac{\partial^2 V}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=L}=0\,, \tag{25}$$

and for both sides free support:

$$w\,|_{\xi^2=r_i}=0\quad\text{and}\quad \left(\frac{\partial^2 w}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=r_i}=0\,, \tag{26}$$

because (16):

$$w^0\,|_{\xi^2=r_i}=0\,,\ \left(\frac{\partial^2 w^0}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=r_i}=0\,,$$

$$V\,|_{\xi^2=r_i}=0\quad \left(\frac{\partial^2 V}{\left(\partial \xi^2\right)^2}\right)|_{\xi^2=r_i}=0\,, \tag{27}$$

where $r_i = 0$ or $r_i = L$ respectively.

The second of the used methods was the finite element method. Its implementation was made using commercial Autodesk Robot Structural Analysis 2011. The model of the composite plate was built using shell elements. This model allowed for the task of boundary conditions of free support on both sides as (25) and bracket as (22−23).

As a third method we used, heuristic method, which is based on a simple, consistent with engineering intuition, averaged mechanical properties of the plate in one direction and bringing the problem to the problem of vibration of one-dimensional beams with variable cross-section.

This approach is equivalent to a very simplified approach using TTA: we use only averaging in the ξ_1 direction and we do not assume a displacement field disjoined (16). The system of equations (21) then reduces to one equation:

$$\left\langle BH^{2222} w^0_{,22}\right\rangle_{,22} + \langle \mu \rangle \ddot{w} = 0 \tag{28}$$

Next we assume harmonic vibrations as:

$$w = \overline{w}\cos(\omega t) \tag{29}$$

We get the equation of vibrations of beams with variable cross-section:

$$\left\langle BH^{2222} w^0_{,22}\right\rangle_{,22} + \omega^2 \langle \mu \rangle w = 0 \tag{30}$$

which could be solved numerically using finite difference method.

3. NUMERICAL RESULTS

For received numerical results we used following materials:

- matrix: E_1 = 20GPa, ν_1 = 0.2, ρ_1 = 2800 kg/m^3
- walls: E_2 = 210GPa, ν_2 = 0.7, ρ_2 = 7800 kg/m^3

and geometrical data:

- microstructure size λ = *0.1 m*
- thickness of plate h = *3 cm,*
- bandwidth L = *1 m,*
- d_2 = *0,*

Below we show some received numerical results:

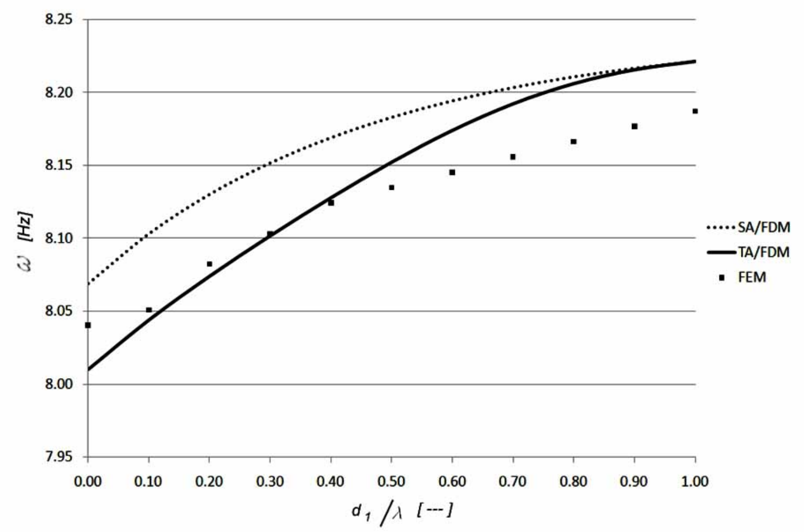

Fig. 2. The results of first frequency of free vibrations for both sides clamped.

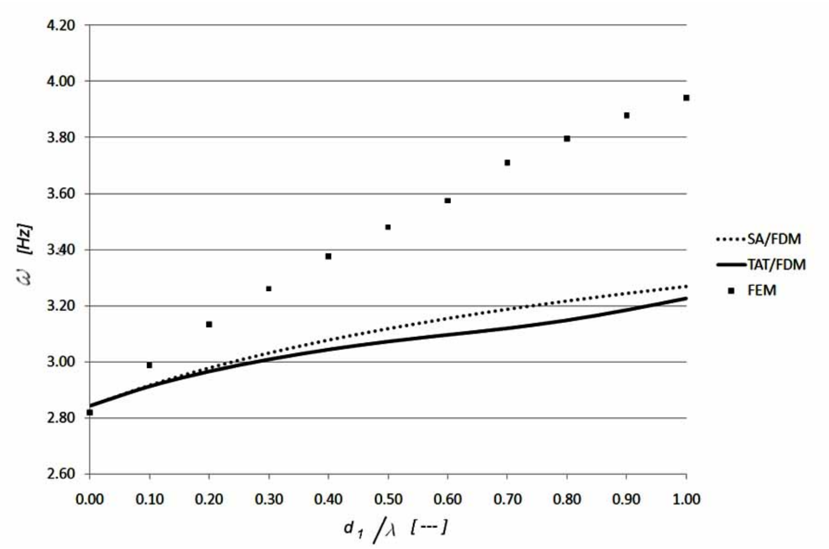

Fig. 3. The results of first frequency of free vibrations for both sides free support.

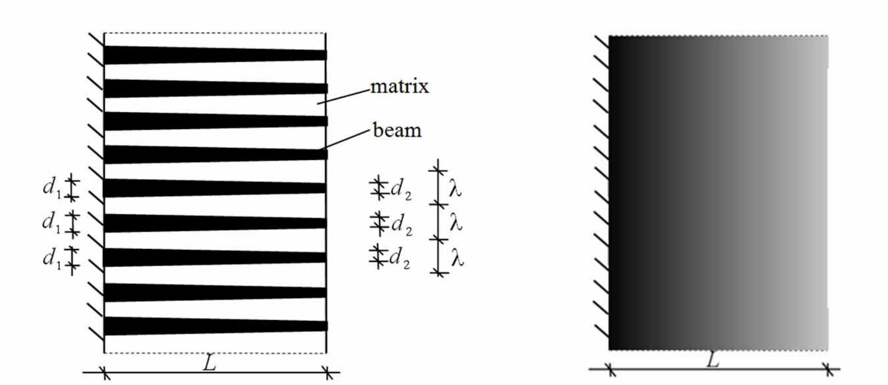

Fig. 4. The results of first frequency of free vibrations for bracket (left side clamped, right side free).

4. CONCLUSIONS

After modelling and analysis received numerical results we can make some conclusions:

1. We can see very strong dependency of frequency of free vibrations of plate band from material distribution for bracket and very weak for both side free support.
2. For cases of symmetric boundary conditions, all three methods give similar results (differences <2%).
3. Differences between the results obtained by the TAT and the SA does not exceed 1.5% for the bracket and 0.5% for the cases of symmetric boundary conditions.
4. The level of complexity (computational effort) is much greater for the TAT method than SA.
5. A very large discrepancy of the results obtained for the bracket between the FEM and the TAT and SA (> 20%) requires further research and comparisons to find the error.

5. ACKNOWLEDGEMENTS

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