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ON THE INFLUENCE OF BOUNDARY LAYER PHENOMENA ONTO AVERAGED TEMPERATURE FIELD

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ABSTRACT

The subject of the contribution is a stationary heat conduction problem in the periodically inhomogeneous rigid conductor. As a tool of modeling the tolerance averaging technique is taken into account, [1]. The aim of the considerations is to reformulate the tolerance averaged model of the considered composites to the form which consists of a single equation for averaged temperature and separated formulas represented a certain solution of the boundary layer equation. The characteristic feature of such form of the tolerance model equations is that the single equation for the averaged temperature field includes an integral operator being a certain generalization of the well-known effective modulus matrix.

Key words: heat conduction, periodically inhomogeneous conductor, boundary effect phenomena

INTRODUCTION

In the most of approaches to the mathematical modeling of the boundary-layer phenomena in the periodic composites the typical investigations are pointed out to formulate an equation or a system of equations in which basic unknowns describing the evolution of boundary perturbations onto the interior of the composite are inde[endent on the averaged temperature field. In the framework of tolerance averaged model, [1,2] this idea leads to the investigation of a possible form of two separations: 1^o from the model system equations a certain part not depending on the averaged temperature or averaged displacements field and 2^o the similar separation of the fluctuation amplitudes from model equations. Such separations are rather impossible over the direct mathematical way without any changing of the mathematical character of the model equations system and that is why various approximations of the tolerance model equations, cf. [3], as well as some new interpretations of constitutive relations have been applied to obtain an independent on the averaged temperature or averaged displacements description of the boundary layer phenomena, cf. [4].

The aim of this paper is to show how to reduce the tolerance model equations for the heat conduction for the micro-periodic composites to the single differential-integral equation with the averaged temperature field as a basic unknown. Then, fluctuation amplitudes should depend on the averaged temperature by direct mathematical formulas. Mentioned above form of the tolerance model equations will be obtained by a certain adaptation of the concept of a variation of constants method, well-known in the classical differential calculus course.

Throughout the paper we shall restrict ourselves to the composites with a periodic two-dimensional hexagonal-type microstructure made of the perfectly bonded constituents. We shall assume that considered composite occupy the region $\Omega = \Xi \times (0, H) \subset R^3$ and every cross-section of the composite which is perpendicular to $0x_3$-axis direction is a sum of a large number of identical regular hexagons, cf. Fig.1. As an additional restriction we shall assume that for any $z \in (0, H)$ the periodicity hexagon is independent on the $2\pi/3$-rotations with respect to the axis parallel to the $0x_3$-axis direction including centers points of the mentioned regular hexagons as axes of the rotations.

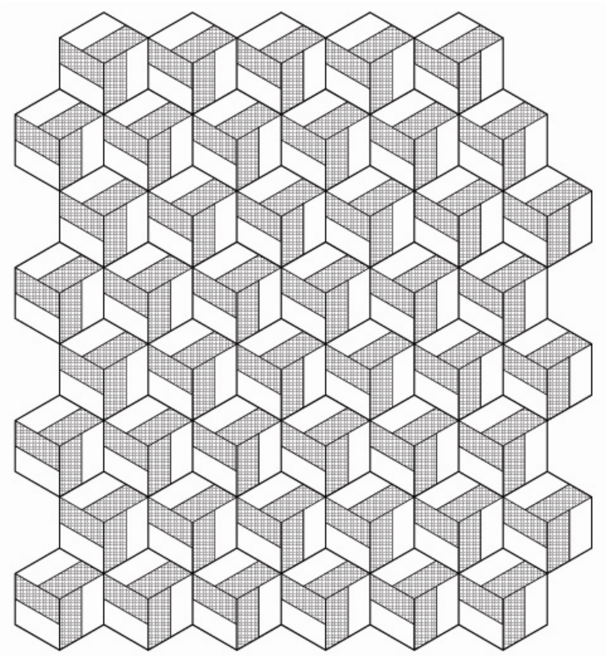

Fig. 1. A cross-section of the two-dimensional periodic composite with internal symmetries

The well-known stationary heat transfer equation based on the Fourier heat conduction law will be taken as the subject of investigation. Under denotations $\nabla = [\partial_1, \partial_1, 0]$, $\partial = [0,0,\partial_3]$ it can be written in the form

$$(\nabla + \partial) \cdot (\mathbf{K} \nabla w) = f \tag{1}$$

Symbol $w = w(\cdot)$ stand here for the temperature field defined in $\Omega = \Xi \times (0, H) \subset R^3$. At the same time by $f = f(\cdot)$ we denote the known density of heat sources, by $c = c(\cdot)$ is the specific heat and

$$\mathbf{K}(x, z) = \begin{bmatrix} \mathbf{A}(x, x_3) & \mathbf{h}^T(x, x_3) \\ \mathbf{h}(x, x_3) & a(x, x_3) \end{bmatrix} \tag{2}$$

is the anisotropic conductivity matrix. Here and in the sequel denotation $x = (x_1, x_2)$ have been applied. Under mentioned above assumptions heat conduction matrix $\mathbf{K} = \mathbf{K}(\cdot, x_3)$ as well as specific heat $c = c(\cdot, x_3)$ are certain fields periodic with respect to vectors $\mathbf{t}^1, \mathbf{t}^2, \mathbf{t}^3$ determining the basic hexagon included in $0x_1x_2$-plane, $\mathbf{t}^1 \cdot \mathbf{t}^2 = \mathbf{t}^2 \cdot \mathbf{t}^3 = \mathbf{t}^3 \cdot \mathbf{t}^1$, $\| \mathbf{t}^1 \| = \| \mathbf{t}^2 \| = \| \mathbf{t}^3 \|$. We shall also assume heat flux continuity conditions in directions normal to interfaces between the constituents of the considered composite.

It is the well-known fact that due to the discontinuous and highly oscillating form of functional coefficients $c = c(\cdot)$, $\mathbf{K} = \mathbf{K}(\cdot)$, the direct applications of (1) to the analysis of special problems in most cases is rather difficult. That is why the mentioned above heat conduction problem is usually replaced, under some additional assumption, by some other problems formulated in the framework of models describing by equations with more regular coefficients. The characteristic feature of these models is that in the framework of which we deal with the microstructure of the considered composite characterized by a certain scalar parameter $\lambda > 0$. In this case conductivity coefficients $c = c(\cdot)$, $\mathbf{K} = \mathbf{K}(\cdot)$, in (1) depend on λ. Since the tolerance averaged model of the considered composite will be taken as the subject of consideration we shall assume that λ is sufficiently small when compared to the characteristic length dimension of the region Ξ. Tolerance model consists of the system of differential equations with constant coefficients and have averaged temperature $u = u(\cdot) \in SV_0^1(\Omega)$ and fluctuation amplitudes $w^A = w^A(\cdot,t) \in SV_0^1(\Omega)$, $A = 1,...,N$, as new basic unknowns. The averaging operation is taken here in the sense of averaged integral operator respect to an arbitrary chosen regular hexagon in $0x_1x_2$ -plane. Introduced above functional space $SV_0^1(\Omega)$ is a certain new space consisting of slowly-varying functions. For particulars the reader is referred to [1,2].

MODEL EQUATIONS

There are known two methods by application of which the tolerance averaged model can be obtained. The first one is the method based of a new concept of *the extended stationary action principle*, cf. [1]. This method has been resulted in many applications dealing functionally graded materials. To the periodic problems *the orthogonalization method* explained in [2] usually have been applied. Since the subject of this paper is a certain periodic material structures we shall restrict ourselves to the tolerance model equations obtained by the last method. Following procedure explained in [2], we look for the temperature field in the form

$$w(x,x_3,t) = u(x,x_3,t) + g^A(x,x_3)W^A(x,x_3,t) \tag{3}$$

where $u(x,x_3,t) = \langle c\rangle^{-1}\langle cw\rangle(x,x_3,t)$ is the averaged temperature field and $W^A(x,x_3,t)$ are extra unknowns which are usually referred to as *the fluctuation amplitudes.* Here and in the sequel $\langle\cdot\rangle$ stand for the integral averaged operator over the basic hexagon, cf. [1,2], and the summation convention holds. Capital Latin superscripts A, B,... run over $1,...,N$, where N is a number of fluctuation amplitudes. *Shape functions* $g^A(x,z)$, $A = 1,...,N$, caused by the material structure of the composite, should be periodic and should satisfy some additional conditions like $\langle g^A\rangle(z) = 0$ and $g^A(\cdot,z) \in O(\lambda)$ for any $z \in (0,H)$, cf. [1,2]. It must be emphasized that the heat flux continuity sassumption in directions normal to interfaces between the constituents imposes on the residual field $g^A(x,x_3)W^A(x,x_3,t)$ in (3) as well as on the material properties of the considered composite additional restrictions. For example if shape functions are piecewise linear then tolerance averaged model holds exclusively not for all conductivuty matrices (2). However, if shape functions are properly chosen, cf. [2], p103, this inconvinience does not impose any restrictions onto conductivity matrix (2). The system of tolerance averaged equations for the stationary heat transfer equation (1) in the considered hexagonal-type composite will be rewritten in the form

$$\begin{aligned}
&\nabla \cdot (\langle \mathbf{A}\rangle \nabla u + \langle a\rangle \partial^2 u + \langle \mathbf{A} \cdot \nabla g^A\rangle \nabla W^A = \langle f\rangle \\
&\langle a g^A g^B\rangle \partial \cdot \partial W^B - \langle \nabla g^A \cdot \mathbf{A} \cdot \nabla g^B\rangle W^B - \\
&-(\langle \nabla g^A \cdot \mathbf{h} g^B\rangle - \langle g^A \mathbf{h} \cdot \nabla g^B\rangle) \cdot \partial W^B - \langle \nabla g^A \cdot \mathbf{A}\rangle \cdot \nabla u = \langle f g^A\rangle
\end{aligned} \tag{4}$$

The above system consists of a single equation describing the evolution of the averaged temperature field u and of N equations describing the evolution of fluctuation amplitudes W^A, $A = 1,...,N$. These equations are conjugated by terms $\langle \mathbf{A} \cdot \nabla g^A\rangle \nabla W^A$ and $\langle \nabla g^A \cdot \mathbf{A}\rangle \nabla u$ in the first and in the second equation, respectively. Under the periodic structure of the composite we shall observe that all averaged coefficients in equations (4) depend on z variable and are independent on x –variable. In the special case in which **h**=**0** model equations (4) reduces to the form presented in [1], p. 102.

To simplify considerations we shall restrict ourselves to the hexagonal-type structures satisfying the following two assumptions:

Assumption 1. The material structure of the anisotropic conductor is invariant under $2\pi/3-$ rotations with respect to the center of a regular periodicity cell.

Assumption 2. The sequence $g^1,...,g^N$ of the shape functions is invariant under $2\pi/3$ – rotations with the center of a regular periodicity cell as the origin of the rotation.

It can be shown that under the above assumptions tolerance equations (3) can be transformed to the equivalent system of equations with the isotopic coefficients, cf. [5,6].

To this end following [6] we shall observe that $N = 3n$ and for a certain positive real n introduce new representation $g^a_{r+1}(x,z) = g^a_r(Qx,z)$, $a = 1,...,n$ of periodic shape functions $g^1,...,g^N$ in which Q denote the $2\pi/3$-rotation with respect to the center of the arbitrary hexagon chosen from the set of basic hexagon. This rotation is the same for every z-variable. At the same time we shall introduce related numbering of fluctuation amplitudes W^A, $A = 1,...,N$, which will be replaced by the sequence W^a_r, $a = 1,...,n$, $r = 1,2,3$. To introduce the alternative form of tolerance model equations (4) we shall also introduce new fluctuation amplitudes which are certain vector fields defined in R^2 and given by

$$\mathbf{v}^a = \mathbf{t}^r W^a_r \tag{5}$$

Formula (5) represent an invertible linear transformation between (W^a_1, W^a_2, W^a_1) and $\mathbf{v}^a$. Let us introduce the following averaged coefficients

$$\begin{aligned}
\mathbf{A}^{ab} &= \langle \nabla g^a_r \cdot \mathbf{A} \cdot \nabla g^b_s \rangle \mathbf{t}_r \otimes \mathbf{t}_s, \ \mathbf{B}^a = \langle \mathbf{A} \cdot \nabla g^a_r \rangle \otimes \mathbf{t}_r, \\
\mathbf{C}^{ab} &= \langle c g^a_r g^b_s \rangle \mathbf{t}_r \otimes \mathbf{t}_s, \quad \mathbf{D}^{ab} = \langle a g^a_r g^b_s \rangle \mathbf{t}_r \otimes \mathbf{t}_s, \\
\mathbf{s}^{ab} &= \langle (\mathbf{t}_r \cdot \nabla g^a_r \cdot \mathbf{K}) g^b_s \rangle \mathbf{t}_s - \langle g^a_r (\mathbf{t}_p \cdot \mathbf{K} \cdot \nabla g^b_p) \mathbf{t}_r \rangle \\
\mathbf{f}^a &= \langle f g^a_r \rangle \mathbf{t}_r,
\end{aligned} \tag{6}$$

where $\mathbf{t}^2 = Q\mathbf{t}^1$, $\mathbf{t}^3 = Q\mathbf{t}^2$, $\mathbf{t}^1 = Q\mathbf{t}^3$, are three vectors determines the basic hexagon and $\tilde{\mathbf{t}}^r \cdot \mathbf{t}^s = \delta^{rs}$ for $r,s = 1,2,3$. Rather simple manipulations yield to

$$\begin{aligned}
&\nabla \cdot (\langle \mathbf{A} \rangle \cdot \nabla u) + \partial \cdot \langle a \rangle \cdot \partial u + \mathbf{B}^a : \nabla \mathbf{v}^a = \langle f \rangle \\
&\mathbf{D}^{ab} \cdot \partial^2 \mathbf{v}^b - \mathbf{s}^{ab} \cdot \partial \mathbf{v}^b - \{\mathbf{A}\}^{ab} : \mathbf{v}^b - (\mathbf{B}^a)^T \cdot \nabla u = \mathbf{f}^a
\end{aligned} \tag{7}$$

where the single dot and the double dot denote single and double contraction of matrices, respectively. Denote by **1** and by $\in$ the unit 2×2 matrix and the Ricci 2×2 matrix, respectively. It can be proved that

$$\langle \mathbf{A} \rangle = k\mathbf{1}, \quad \{\mathbf{A}\}^{ab} = \hat{a}^{ab}\mathbf{1} + \breve{a}^{ab} \in, \ \mathbf{B}^a = \hat{b}^a \mathbf{1} + \breve{b}^a \in, \ \mathbf{D}^{ab} = \hat{d}^{ab}\mathbf{1} + \breve{d}^{ab} \in, \tag{8}$$

for

$$\begin{aligned}
&k = 0.5\langle tr\mathbf{A} \rangle, \ \hat{b}^a = \frac{3}{4}\langle (tr\mathbf{A}) \nabla g^a_r \rangle \otimes \mathbf{t}^r, \ \breve{b}^a = \frac{3}{4}\langle (tr\mathbf{A}) \nabla g^a_r \rangle \otimes \mathbf{t}_r, \\
&\hat{a}^{ab} = \mathbf{t}^r \cdot \langle \nabla g^a_r (tr\mathbf{A}) \nabla g^b_s \rangle \cdot \mathbf{t}^s, \ \breve{a}^{ab} = \frac{3}{8}\mathbf{t}^r \cdot \langle \nabla g^a_r (tr\mathbf{A}) \nabla g^b_s \rangle \cdot \mathbf{t}_s, \\
&\hat{d}^{ab}_2 = \langle a g^a_r g^b_s \rangle \delta^{rs}, \ \breve{d}^{ab} = \langle a g^a_r g^b_s \rangle \in^{rs},
\end{aligned} \tag{9}$$

As an important remark note that since $\mathbf{s}^{ab} = -\mathbf{s}^{ba}$ for any $a,b = 1,...,n$, we have $\mathbf{s} = \mathbf{s}^{11} = -\mathbf{s}^{11} = 0$ and hence for n=1 equations (7) are isotropic. That is why we restrict considerations to the case in which $n = 1$ i.e. in which only three shape functions are taken into account. These shape functions are generated by a certain basic shape function $g(x,z)$ by conditions $g_1(x,z) = g(x,z)$, $g_2(x,z) = g(Qx,z)$, $g_3(x,z) = g(Q^2x,z)$. In this case model equations (9) can be rewritten in the form

$$\begin{aligned}
&\nabla \cdot (\langle \mathbf{A} \rangle \cdot \nabla u) + \mathbf{B} : \nabla \mathbf{v} = \langle f \rangle \\
&\mathbf{D}\partial^2 \mathbf{v} - \{\mathbf{A}\}\mathbf{v} = \mathbf{f} + \mathbf{B}^T \nabla u
\end{aligned} \tag{10}$$

The above model equations are starting point for the subsequent considerations.

FLUCTUATION AMLITUDE DECOMPOSITION

Now, we are to decompose fluctuation amplitude **v** onto the sum of two terms

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \tag{11}$$

and hence the first from tolerance model equations (10) can be rewritten in the form

$$\nabla \cdot (\langle \mathbf{A} \rangle \nabla u) + \langle a \rangle \partial^2 u = \langle f \rangle - \mathbf{B} : \nabla \mathbf{v}_0 - \mathbf{B} : \nabla \mathbf{v}_1 \tag{12}$$

The first term, denoted by $\mathbf{v}_0(x,z)$, should be identified with the whole family of integrals of ordinary differential equation

$$\mathbf{D}\partial^2 \mathbf{v}_0 - \{\mathbf{A}\}\mathbf{v}_0 = 0 \tag{13}$$

which will be referred to as *a boundary layer equation*. For an arbitrary quadratic matrix $\mathbf{X}$ denote by $e^{\mathbf{X}} = \sum_{n=0}^{n=+\infty} \frac{\mathbf{X}^n}{n!}$ *the exponential of matrix* $\mathbf{X}$, cf. [8], and, in the case in which matrix $\mathbf{X}$ is positive definite, by $\sqrt{\mathbf{X}}$ unique positive definite quadratic matrix satisfying condition $\sqrt{\mathbf{X}}^2 = \mathbf{X}$, [9]. It is easy to verify that boundary layer equation (13) has a solution $\mathbf{v}_0 = e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}\mathbf{c}_1 + e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}\mathbf{c}_2$, where $\mathbf{c}_1, \mathbf{c}_2 \in R^2$ are arbitrary vectors. Let us assume boundary conditions in the form

$$\mathbf{v}_0(x,0) = \mathbf{q}_0(x), \quad \mathbf{v}_0(x,0) = \mathbf{q}_H(x), \tag{14}$$

These boundary conditions yield

$$\begin{aligned} \mathbf{q}_0 &= \mathbf{c}_1 + \mathbf{c}_2 \\ \mathbf{q}_H &= \mathbf{c}_1 e^{-H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} + \mathbf{c}_2 e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \end{aligned} \tag{15}$$

Hence

$$\begin{aligned} \mathbf{c}_1 &= \frac{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1}\mathbf{q}_0 - \frac{1}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1}\mathbf{q}_H \\ \mathbf{c}_2 &= -\frac{1}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1,}\mathbf{q}_0 + \frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1}\mathbf{q}_H \end{aligned} \tag{16}$$

Finally the solution to boundary layer equation (13) for boundary conditions (14) is given by

$$\begin{aligned} \mathbf{v}_0 = &\left(\frac{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\mathbf{A}}} - 1}\mathbf{q}_0 - \frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\mathbf{A}}} - 1}\mathbf{q}_H \right) e^{-\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3} + \\ &+ \left(-\frac{1}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1,}\mathbf{q}_0 + \frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} - 1}\mathbf{q}_H \right) e^{\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3} \end{aligned} \tag{17}$$

The second term, denoted by $\mathbf{v}_1(x,z)$, should be a certain special solution to the ordinary differential equation

$$\mathbf{D}\partial^2 \mathbf{v}_1 - \{\mathbf{A}\}\mathbf{v}_1 = \mathbf{B}^T \nabla u + \langle fg \rangle \tag{18}$$

This solution will be investigated in the form

$$\mathbf{v}_1(x,x_3) = e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}\mathbf{c}_1(x,x_3) + e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}\mathbf{c}_2(x,x_3) \tag{19}$$

In which $\mathbf{c}_1(x,x_3)$ and $\mathbf{c}_2(x,x_3)$ should be determined from

$$\begin{bmatrix} e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} & e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \\ -\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} & \sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \end{bmatrix}\begin{bmatrix}\mathbf{c}_1 \\ \mathbf{c}_2\end{bmatrix} = \begin{bmatrix}\mathbf{0} \\ \mathbf{B}^T\nabla u+\mathbf{f}\end{bmatrix} \tag{20}$$

It is the well-known fact that, under the Liouville lemma, the Wroński matrix taken as the basic matrix for the above equation, is invertible and hence (20) has unique solution

$$\begin{gathered}\begin{bmatrix}\mathbf{c}_1 \\ \mathbf{c}_2\end{bmatrix} = \begin{bmatrix} e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} & e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \\ -\sqrt{\mathbf{A}^{-1}\mathbf{D}}e^{-x_3\sqrt{\mathbf{D}^{-1}\mathbf{A}}} & \sqrt{\mathbf{A}^{-1}\mathbf{D}}e^{x_3\sqrt{\mathbf{D}^{-1}\mathbf{A}}} \end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0} \\ \mathbf{B}^T\nabla u+\mathbf{f}\end{bmatrix} = \\ = \frac{1}{2}\begin{bmatrix} e^{x_3\sqrt{\mathbf{D}^{-1}\mathbf{A}}} & -\sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}e^{x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \\ e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} & \sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \end{bmatrix}\begin{bmatrix}\mathbf{0} \\ \mathbf{B}^T\nabla u+\mathbf{f}\end{bmatrix}\end{gathered} \tag{21}$$

Hence

$$\begin{gathered}\mathbf{v}_1(x,x_3) = \mathbf{v}_0 + \\ +\frac{1}{2}\int_{z=z_0}^{z=x_3} [e^{-x_3\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}, e\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}]\begin{bmatrix} e^{z\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} & -\sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}e^{z\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \\ e^{-z\sqrt{\mathbf{D}^{-1}\mathbf{A}}} & \sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}e^{-z\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}} \end{bmatrix}\begin{bmatrix}0 \\ \mathbf{f}+(\mathbf{B}^T)\nabla u\end{bmatrix}dz\end{gathered} \tag{22}$$

and finally solution to the ordinary differential equation (13) for boundary conditions (14) is given by

$$\begin{gathered}\mathbf{v}(x,x_3) = \left(\frac{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\mathbf{A}}}-1}\mathbf{q}_0 - \frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\mathbf{A}}}-1}\mathbf{q}_H\right)e^{-\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3} + \\ +\left(-\frac{1}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1,}\mathbf{q}_0 + \frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1}\mathbf{q}_H\right)e^{\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3} - \\ -\int_{z=z_0}^{z=x_3} \sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}[\sinh\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\ z](\mathbf{f}+\mathbf{B}^T\nabla u)dz\end{gathered} \tag{23}$$

where $\sinh\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\ z = \frac{1}{2}(e^{\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\,z} - e^{-\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\,z})$.

Now we are to introduce the concepts of *the residual operator* and *the effective operator* denoted by $R[\cdot],\ A^{eff}[\cdot]$, respectively, which are defined by

$$\begin{gathered}R[\nabla u](x,x_3) = \sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}\int_{z=z_0}^{z=x_3} (\sinh\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\ z)\,\nabla^2 u(x,z)dz \\ A^{eff}[u](x,x_3) = \nabla\cdot(\langle\mathbf{A}\rangle\nabla u)(x,x_3) - \mathbf{B}:\nabla R[\mathbf{B}^T\nabla u](x,x_3)\end{gathered} \tag{24}$$

Under the above denotations we have

$$\begin{gathered}A^{eff}[u](x,x_3) = \nabla\cdot(\langle\mathbf{A}\rangle\nabla u)(x,x_3) - \\ -\mathbf{B}:\left\{\sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}\int_{z=z_0}^{z=x_3} (\sinh\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\ z)\nabla^2 u(x,z)dz\right\}\mathbf{B}^T\end{gathered} \tag{25}$$

and equation (13) can be rewritten in the form independent on the fluctuation amplitudes

$$A^{eff}[u] + \langle a\rangle\partial^2 u = \langle f\rangle - \mathbf{B}:\nabla R[\mathbf{f}] - \nabla\mathbf{v}_0 \tag{26}$$

where $\mathbf{v}_0$ is the solution of boundary layer equation given by (17). In the special case in which solution $\mathbf{v}_0$ to the boundary layer equation (13) vanish (for example if boundary conditions (14) are homogeneous, $\mathbf{q}_0 = \mathbf{q}_H = \mathbf{0}$) and $\mathbf{f} = \mathbf{0}$, equation (26) takes the simpler form

$$A^{eff}[u]+\langle a\rangle\partial^2 u=\langle f\rangle \tag{27}$$

Bearing in mind that temperature perturbations imposed by the on the boundaries $z=0$ and $z=H$ are muffled in the points placed not close to the boundary of the region occupied by the composite it must be supposed that in many points of Ω the approximation off effective operator $A^{eff}[\cdot]$ based on the integral mean value property can be applied:

$$\begin{aligned} A^{eff}[\nabla u](x,x_3) &\approx \nabla\cdot(\langle\mathbf{A}\rangle\nabla u)(x,x_3)- \\ &\mathbf{B}:\left\{\sqrt{\mathbf{DA}}^{-1}\int_{z=z_0}^{z=x_3}[\sinh\sqrt{\mathbf{D}^{-1}\mathbf{A}}\,z]dz\right\}\mathbf{B}^T\nabla^2 u(x,z_0)\end{aligned} \tag{28}$$

where z_0 is an arbitrary point taken from interval $(0,H)$. Bearing in mind that

$$\int_{z=z_0}^{z=x_3}\sqrt{\mathbf{D}\{\mathbf{A}\}}^{-1}[\sinh\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}\,z]dz=\{\mathbf{A}\}^{-1}$$

the mentioned above approximation of effective operator $A^{eff}[\cdot]$ yield

$$A^{eff}[\nabla u](x,x_3)\approx\nabla\cdot(\langle\mathbf{A}\rangle-\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T)\nabla u(x,z_0)) \tag{29}$$

Hence, under denotation

$$\mathbf{A}^{eff}=\langle\mathbf{A}\rangle-\mathbf{B}\{\mathbf{A}\}^{-1}\mathbf{B}^T]$$

equation (27) take the form

$$\nabla\cdot(\mathbf{A}^{eff}\nabla u)(x,z_0)+\langle a\rangle\partial^2 u(x,x_3)=\langle f\rangle \tag{30}$$

A conclusion that averaged temperature field u should satisfy equation (26) is a basic result of this paper.

FINAL REMARKS

At the end of the paper we are to resume the basic results of the paper in the form of the collecting of just obtained formulas to the following alternative system of tolerance model equations

$$\begin{aligned} &A^{eff}[\nabla^2 u]+\langle a\rangle\partial^2 u-\langle c\rangle\partial_t u=\langle f\rangle-\mathbf{B}:\nabla\cdot[\mathbf{f}]-\nabla\mathbf{v}_0 \\ &\mathbf{v}_0=\left(\frac{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1}\mathbf{q}_0-\frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1}\mathbf{q}_H\right)e^{-\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3}+ \\ &+\left(-\frac{1}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1,}\mathbf{q}_0+\frac{e^{H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}}{e^{2H\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}}-1}\mathbf{q}_H\right)e^{\sqrt{\mathbf{D}^{-1}\{\mathbf{A}\}}x_3}\end{aligned} \tag{31}$$

which consists of the single equation (26) for the averaged temperature u and formula (17) including as a first term the solution to the boundary layer equation (13). It must be emphasized that the averaged temperature field depends on the boundary layer phenomena by the additional source term equal to $-\nabla\mathbf{v}_0$. This term vanish provided that boundary conditions (14) are independent on $x\in\Xi$.

As a special remark we shall pointed out that in most applications of the tolerance modeling technique we deal with exclusively one shape function, cf. [10,11,12,13,14] and hence applied in this paper reducing of tolerance model equations to the case with one fluctuation amplitude should be treated as a very important property of the described physical situation. However, mentioned above papers deal non-stationary cases of heat conduction problems or dynamic problems in elasticity. In this cases described in the paper reducing of tolerance model equations to the single equation for the averaged temperature field seems to be also possible to realization. It will be explained in the separate paper.

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