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ON THE LENGTH-SCALE EFFECT IN THE FREE VIBRATIONS OF PERIODICALLY RIBBED ELASTIC PLATES

Katarzyna Jeleniewicz, Wiesław Nagórko

Department of Civil Engineering and Geodesy, Warsaw University of Life Sciences – SGGW, Poland

ABSTRACT

The plates reinforced by ribs are being considered in the paper. For such plates, by assuming a periodic distribution of the ribs in the plate, the average model was being constructed, consisting the plate dynamics equations, which are the linear differential ones with constant coefficients. For modeling the tolerance not asymptotic averaging technique were used. Consequently, in the modeling equations, a microstructure parameter remains (basic cell size). In the paper the free vibrations of the consideration plates were tested and the presence of scale effect was demonstrated.

Key words: ribbed plate, elastodynamics of the plates, vibrations of the plates, non-asymptotic homogenization

INTRODUCTORY CONCEPTS

The plates considered in the paper are peculiar structural elements, in which it is possible to distinguish a smooth surface in the range several times greater than the thickness of these elements. The term thickness is being used here as the maximum dimension of structural element, measured in the normal direction to the highlighted surface.

The role of the plates in the construction industry is huge, so it becomes necessary to construct adequate theories, which are a convenient basic for design.

The plates are three-dimensional objects, and if the material is elastic, they can be described and studied at the base of the theory of elasticity. Solving initially-boundary problems in the framework of this theory is very complicated and therefore a simpler theory was being looked for (in the case of the plates – two-dimensional theory, also known as surface one).

In the two-dimensional theory the displacements and stresses describe functions depending only on two variables (in physical space) and are defined in the above-mentioned highlighted surface – *middle surface*. In the three-dimensional theory the displacements and stresses depend on three spatial variables. A closer analysis of this model is put as an example in the work [1].

The history of creating the theory of plate is very long. As the absolute beginning it may be considered the experimental work by E. F. F. Chladni (1756-1827), under which J. Bernoulli in 1789 published a paper, in which he formulated the amplitude equation of the plate vibrations. This equation, however, was not correct.

After many attempts by various authors, only G. Kirchhoff in 1850 obtained an equation describing the deflection of thin plates, and his theory is still in use. The description of the historical formation and development of the theory of plates and shells can be found in the monograph [2].

REINFORCED PLATES

The object of considerations are rectangular elastic plates reinforced by periodically spaced ribs (Fig. 1).

The aim of the work is to build the easier model for such plates. in which the system of equations will be differential ones with constant coefficients. Then, the analysis of free vibrations of the plate which may present the existence the scale effect in the model, i.e. demonstrate the solutions dependence on the dimension of a basic cell.

It is assumed that the plate are subjected to the plane stress. A configuration of the plate will be a region $\Pi = (-L, L) \times (-H, H)$, $(x_1, x_2) \in \Pi$. If $L \to \infty$ one will deal with a plate-band. By I one denotes known time interval, $t \in I = < t_0, t_1 > \subset R$. The plate will be reinforced by the ribs of thickness l' and l'', spaced alternately and parallel to x_2-axis. The distances between the ribs are equal.

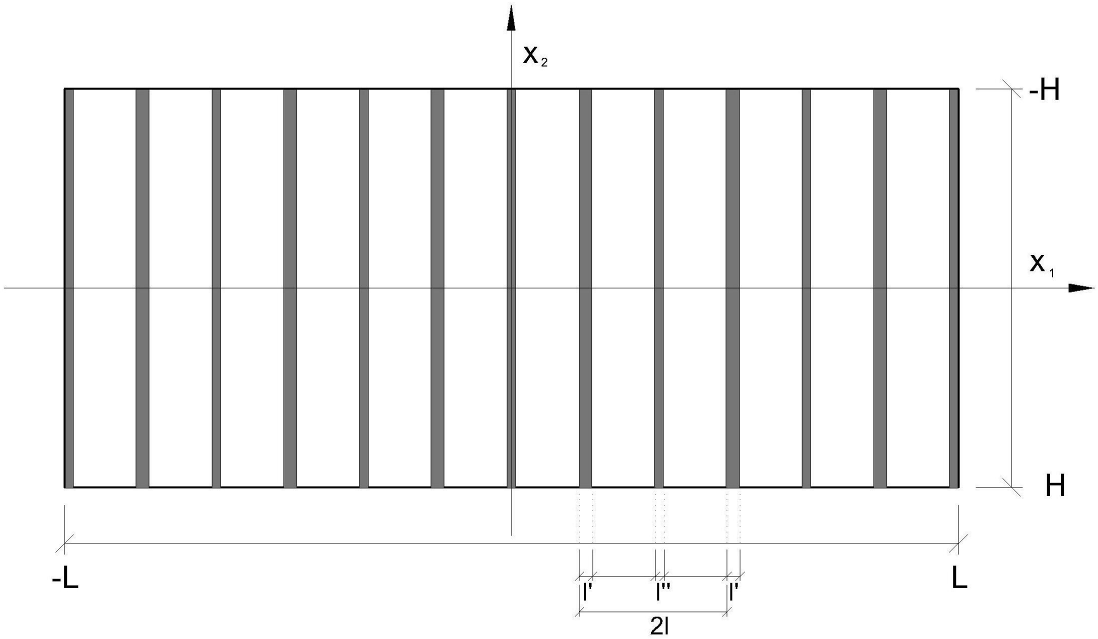

Fig. 1. Rectangular elastic plates reinforced by periodically spaced ribs

Let $2l$ be a length of the basic cell denoted by $\Delta = (-l, l)$. In that case the distances between the ribs are equal $l - \frac{l' - l''}{2}$. It may be assumed that $l' + l'' << 2l$.

The plate material is homogeneous and isotropic with Lamė module λ, μ and mass density M_0. Under the plate stress assumption instead of modulus λ the reduced modulus $\lambda_0 \equiv \frac{2\lambda\mu}{1 + 2\mu}$ was introduced.

The ribs are assumed to be slender in Ox_1x_2 – plate and carried out only axial stress. Hence their properties are determined by Young modulus E_α and mass density M_α, $\alpha = 1{,}2$. Moreover the thickness of the ribs may be neglected as small if compared to the distance l between the ribs axes. At the same time distance l is very small with respect to length dimension, L, $l << L$ as it was shown in Fig 1.

We denote by $w_\alpha(x_1, x_2, t)$, $\alpha = 1{,}2$, $x_1 \in (-L, L)$, $x_2 \in (-H, H)$, $t \in R$ components of the displacement vector field in Ox_1x_2 –plane, (the total plate deflection).

In the case of heterogeneous plates, particularly reinforced, the equations describing the dynamic plane problems, in the theory of elasticity, have the form

$$\rho(x_1)\ddot{w}_\alpha(x_1, x_2, t) - [C_{\alpha\beta\gamma\delta}(x_1) w_{\gamma,\delta}(x_1, x_2, t)]_{,\beta} = 0 \tag{1}$$

where $C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}(x_1, x_2)$, $\alpha, \beta, \gamma, \delta = 1,2$ are materials functions of the plate, and $\rho = \rho(x_1)$ is mass density related to the Ox_1x_2 -plane. In the equation (1) $\ddot{w}_\alpha$ marked the second derivative of the time, and used designation for partial derivative function *f* on spatial variable x_α in the form $f_{,\alpha} \equiv \dfrac{\partial f}{\partial x_\alpha}$. In this work is in force summation convention.

In the case of plates reinforced periodically, functions $C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}(x_1, x_2)$ occurring in the equations (1) are discontinuous functions, stepwise changing values. Solving these equations, including numerical, is very difficult.
The boundary conditions for a plate are assumed in the form

$$w_1(\pm L, x_2, t) = 0, \quad w_2(x_1, \pm H, t) = 0, \quad (\partial_2 w_1 + \partial_1 w_2)(x_1, \pm H, t) = 0, \tag{2}$$

For the plate-band the boundary conditions on $x_1 = \pm L$ are neglected.
Introducing the Dirac functions $\delta(\cdot)$ of argument $x_1 \in R$ and setting, [5];

$$\widetilde{M}_\alpha(x_1) = M_\alpha \delta(x_1 \pm l_\alpha)$$

$$\widetilde{E}_\alpha(x_1) = E_\alpha \delta(x_1 \pm l_\alpha), \quad \alpha = 1,2,...$$

where $l_1 = (2n - l)l$, $l_2 = 2nl$, $n = 0, \pm 1, \pm 2, ...$, it can be defined

$$\rho(x_1) = M_0 + \widetilde{M}_1(x_1) + \widetilde{M}_2(x_1)$$

$$C_{\alpha\beta\gamma\delta}(x_1) = \lambda_0 \delta_{\alpha\beta}\delta_{\gamma\delta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) + (\widetilde{E}_1 + \widetilde{E}_2)\delta_{\alpha 2}\delta_{\beta 2}\delta_{\gamma 2}\delta_{\delta 2} \tag{3}$$

By substituting the functions (3) into the equation (1) one obtains three-dimensional description of the ribbed plates in which there are variable coefficients, stepwise changing values in small intervals of definiteness. This causes a significant obstacle in solving boundary value problems for plates. Continue to construct the model, in which right dynamics equations of ribbed plate will have constants coefficients.

AVERAGED MODEL

The modeling procedure begins with the well known form of Lagrangian for the plane stress problem under consideration

$$L = \frac{1}{2}\rho \dot{w}_\alpha \dot{w}_\alpha - \frac{1}{2} C_{\alpha\beta\gamma\delta} w_{\alpha,\beta} w_{\gamma,\delta} \tag{4}$$

According to the tolerance averaging technique one assumes the decomposition of displacement fields in a form

$$w_\alpha(x_1, x_2, t) = u_\alpha(x_1, x_2, t) + h(x_1) v_\alpha(x_1, x_2, t) \tag{6}$$

where $u_\alpha(\cdot, x_2, t)$ and $v_\alpha(\cdot, x_2, t)$ are slowly-varying functions, which are new looking function, while the function $h(x_1)$ are known $2l$ -periodic and fluctuations shape functions, in the form given in Fig. 2. Functions u_α are the averaged deflection, while functions v_α, called fluctuation, describing the impact of the microstructure (ribbing) plate.

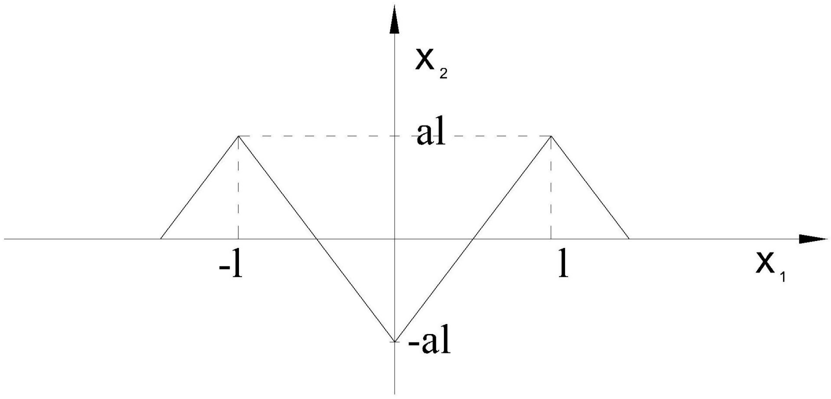

Fig. 2. Fluctuation shape function $h(\cdot)$

Substitute the displacement fields (6) to the functional (4) and averaging this functional, one obtains

$$< L >= \frac{1}{2}[< \rho > \dot{u}_\alpha \dot{u}_\alpha + < \rho h^2 > \dot{v}_\alpha \dot{v}_\alpha + < C_{\alpha\beta\gamma\delta} > u_{\alpha,\beta} u_{\gamma,\delta} + 2 < C_{\alpha\beta\gamma 1} h' > u_{\alpha,\beta} v_\gamma +$$

$$+ < C_{\alpha 2\beta 2} h^2 > v_{\alpha,2} v_{\beta,2} + < C_{\alpha 1\beta 1} (h')^2 > v_\alpha v_\beta] \tag{7}$$

where $h' = \partial h / \partial x_1$ and the averaging operator, for the every integrable functions f is defined as, [3,4]:

$$< f >= \frac{1}{2l} \int_{-l}^{l} f(x_1) dx_1$$

Search the tolerance model equations are the Euler-Lagrange equations functional (7):

$$< \rho > \ddot{u}_\alpha - < C_{\alpha\beta\gamma\delta} > u_{\gamma,\beta\delta} + < C_{\alpha\beta\gamma 1} h' > v_{\gamma,\beta} = 0 \tag{8}$$

$$< \rho h^2 > \ddot{v}_\alpha - < C_{\alpha 2\beta 2} h^2 > v_{\beta,22} + < C_{\alpha 1\beta\gamma} h' > u_{\beta,\gamma} + < C_{\alpha 1\beta 1} (h')^2 > v_\beta = 0$$

It can be seen that the above modeling approach leads from two equations (5) with variable coefficients to the four equations (8) for the averaged deflection $u_\alpha(\cdot)$ and fluctuation variable $v_\alpha(\cdot)$, $\alpha = 1,2$. Equations (8) have constant coefficients and hence constitute the proper mathematical tool for the analysis of special problems. It has to emphasized that solutions to the boundary-value problems for the above equations have a physical sense only if $u_\alpha(\cdot, x_2, t)$, $v_\alpha(\cdot, x_2, t)$ are slowly varying functions,

$$u_\alpha(\cdot, x_2, t) \in S_\delta^1 V(\Delta), \quad v_\alpha(\cdot, x_2, t) \in SV_\delta^1(\Delta), \tag{9}$$

for every $x_2 \in (-H, H)$ and every time t .

Constructed model of a reinforced plate, is non-asymptotic model and is described by the system of equations (8) for the unknown functions $u_\alpha(\cdot)$ and $v_\alpha(\cdot)$, which should satisfy the conditions (9). It should be noted that the constant coefficients in the equations (8) depend on fluctuation shape function $h(x_1)$, which should be known. In these equations there is a parameter *l*, describing the influence of microstructure on the properties of the plate – one can explore the scale effect by use of that model.

ANALYSIS

Let consider the following special case, assuming that

$$\begin{aligned} u_1 &= u_1(x_1, t), \;\; v_1 = v_1(x_1, t), \;\; x_1 \in [-L, L] \\ u_2 &= u_2(x_2, t), \;\; v_2 = v_2(x_2, t), \;\; x_2 \in [-H, H] \end{aligned} \tag{10}$$

Taking into account the terms (10) in the equations (8) one obtains

$$< \rho > \ddot{u}_1 - < C_{1111} > u_{1,11} + < C_{1111} h' > v_{1,1} + < C_{1212} h' > v_{2,2} = 0$$

$$< \rho h^2 > \ddot{v}_1 - < C_{1111} h' > u_{1,1} + < C_{1122} h' > u_{2,2} + < C_{1111} (h')^2 > v_1 = 0$$

$$< \rho > \ddot{u}_2 - < C_{2222} > u_{2,22} = 0$$

$$< \rho h^2 > \ddot{v}_2 - < C_{2222} h^2 > v_{2,22} + < C_{1212} (h')^2 > v_2 = 0 \tag{11}$$

The function shown in Fig. 2 has the following form

$$h(x_1) = \begin{cases} -2x_1 - l & x_1 \in < -l, 0 > \\ 2x_1 - l & x_1 \in < 0, l > \end{cases} \tag{12}$$

In this case the coefficients in the equations (8) are equal

$$< \rho >=< M_0 + \widetilde{M}_1 + \widetilde{M}_2 >= M_0 + \frac{M_1 + M_2}{2l},$$

$$< C_{1111} >=< \lambda_0 + 2\mu >= \lambda_0 + 2\mu ,$$

$$< C_{2222} >=< \lambda_0 + 2\mu + \widetilde{E}_1 + \widetilde{E}_2 >= \lambda_0 + 2\mu + \frac{E_1 + E_2}{2l},$$

$$< \rho h^2 > = < (M_0 + \widetilde{M}_1 + \widetilde{M}_2) h^2 > = l^2 (\frac{M_0}{3} + \frac{(M_1 + M_2)}{2l}) , \tag{13}$$
$$< C_{2222} h^2 > = < (\lambda_0 + 2\mu + \widetilde{E}_1 + \widetilde{E}_2) h^2 > = l^2 (\frac{(\lambda_0 + 2\mu)}{3} + \frac{(E_1 + E_2)}{2l})$$
$$< C_{1111} h' > = < (\lambda_0 + 2\mu) h' > = 0 ,$$
$$< C_{1212} h' > = < \mu h' > = 0 ,$$
$$< C_{1122} h' > = < \lambda_0 h' > = 0 ,$$
$$< C_{1111} (h')^2 > = < (\lambda_0 + 2\mu)(h')^2 > = 4(\lambda_0 + 2\mu) ,$$
$$< C_{1212} (h')^2 > = < \mu (h')^2 > = 4\mu .$$

Substituting the coefficients (13) into the equations (8) one arrives at the system of two independent equations for $u_1(x_1, t)$ i $v_1(x_1, t)$, $x_1 \in [-L, L]$

$$(M_0 + \frac{M_1 + M_2}{2l}) \ddot{u}_1 - (\lambda_0 + 2\mu) u_{1,11} = 0$$
$$l^2 (\frac{M_0}{3} + \frac{M_1 + M_2}{2l}) \ddot{v}_1 + 4(\lambda_0 + 2\mu) v_1 = 0 \tag{14}$$

and two independent equations for $u_2(x_2, t)$ $v_2(x_2, t)$, $x_2 \in [-H, H]$

$$(M_0 + \frac{M_1 + M_2}{2l}) \ddot{u}_2 - (\lambda_0 + 2\mu + \frac{E_1 + E_2}{2l}) u_{2,22} = 0$$
$$l^2 (\frac{M_0}{3} + \frac{M_1 + M_2}{2l}) \ddot{v}_2 - l^2 (\frac{\lambda_0 + 2\mu}{3} + \frac{E_1 + E_2}{2l}) v_{2,22} + 4\mu v_2 = 0 \tag{15}$$

Note that the boundary conditions (2) will now have the form

$$u_1(\pm L, t) = 0, \; v_1(\pm L, t) = 0 , \quad u_2(\pm H, t) = 0, \; v_2(\pm H, t) = 0 \tag{16}$$

Passing to analysis of the equations $(15)_2$. This equation can be rewrite in the form

$$\ddot{v}_2 - A_2 v_{2,22} + B_2 v_2 = 0$$

where

$$A_2 = \frac{2l(\lambda_0 + 2\mu) + 3(E_1 + E_2)}{2lM_0 + 3(M_1 + M_2)} , \; B_2 = \frac{24\mu}{(2lM_0 + 3(M_1 + M_2))l}$$

Setting $v_2(x_2, t) = \psi(x_2)\xi(t)$ (where $\xi(t) = \cos \omega t$) one obtains

$$A\psi'' - (B - \omega^2)\,\psi = 0 \tag{17}$$

The following special cases shall be considered.
If $B > \omega^2$, than

$$\psi'' - \kappa^2 \psi = 0$$

where $\kappa^2 = \frac{B - \omega^2}{A}$. Hence $\psi = C_1 e^{-\kappa x_2} + C_2 e^{\kappa x_2}$. Including the condition $(16)_4$ one obtains $C_1 = C_2 = 0$.

If $B < \omega^2$, than denoting $\kappa^2 = \frac{\omega^2 - B}{A}$ one obtains

$$\psi'' + \kappa^2 \psi = 0$$

It follows that $\psi = C_1 \cos \kappa x_2 + C_2 \sin \kappa x_2$.

Taking in the last solutions $C_2 = 0$, or assuming the solution $\psi = C_1 \cos \kappa x_2$, one obtains $\kappa = \kappa_n = \frac{\pi}{2H} + \frac{n\pi}{H}$, $n = 0, \pm 1, \pm 2, ...$ and $\omega_n = -A_2 \kappa_n^2 + B_2$. Finally

$$v_2(x_2, t) = C_n \cos \kappa_n x_2 \cos \omega_n t \tag{18}$$

When one denotes by $A_1 = \frac{2l(\lambda_0 + 2\mu) + (E_1 + E_2)}{2lM_0 + M_1 + M_2}$, the equation $(15)_1$ takes the form $\ddot{u}_2 - A_1 u_{2,22} = 0$. Looking for solutions in the form analogous to the previously $u_2(x_2, t) = \psi(x_2) \cos \omega t$ one obtains $\psi = C_1 \cos \kappa x_2 + C_2 \sin \kappa x_2$ where $\kappa^2 = \tilde{\omega}^2 / A_1$. Assuming $C_2 = 0$, from the condition $(16)_3$ one determines the free vibrations $\tilde{\omega}_n^2 = A_1 \kappa_n^2$, $n = 0, \pm 1, \pm 2, ...$ and finally one obtains

$$u_2(x_2, t) = C_n \cos \kappa_n x_2 \cos \tilde{\omega}_n t \tag{19}$$

Proceed similarly and by the use of the conditions $(13)_{1-2}$ solutions of the equations (11) takes the form

$$\begin{aligned} u_1(x_1, t) &= C_n \cos \bar{\kappa}_n x_1 \cos \bar{\omega}_n t, \\ v_1(x_1, t) &= C_n (x_1^2 - L^2) \cos \sqrt{B_0}\, t \end{aligned} \tag{20}$$

where $\bar{\kappa}_n^2 = \bar{\omega}_n^2 / A_0$, $\bar{\omega}_n = A_0 (\frac{\pi}{2L} + \frac{n\pi}{L})$, $n = 0, \pm 1, \pm 2, ...$,

$A_0 = \frac{2l(\lambda_0 + 2\mu)}{2lM_0 + M_1 + M_2}$, $B_0 = \frac{24(\lambda_0 + 2\mu)}{l(2lM_0 + 3(M_1 + M_2))}$.

To plot graphs of the analytical solutions (18) - (20) one assumed the material's constants of the plate and the ribs equal $\mu^M = 11{,}25GPa$, $\lambda^M = 7{,}5GPa$, $E^R = E_1 = E_2 = 200GPa$, the plate density and the ribs density were assumed equal $M^M = M_0 = 2400 \frac{kg}{m^3}$, $M^R = M_1 = M_2 = 7900 \frac{kg}{m^3}$.
The dimensions of the plate were as follow $L = 5m$, $H = 5m$, $l = 0{,}3m$.
Figure 3 shows a diagram of the averaged displacement $u_1 = u_1(x_1, t)$ at the first free frequency. Diagram of function $u_2 = u_2(x_2, t)$ is similar.

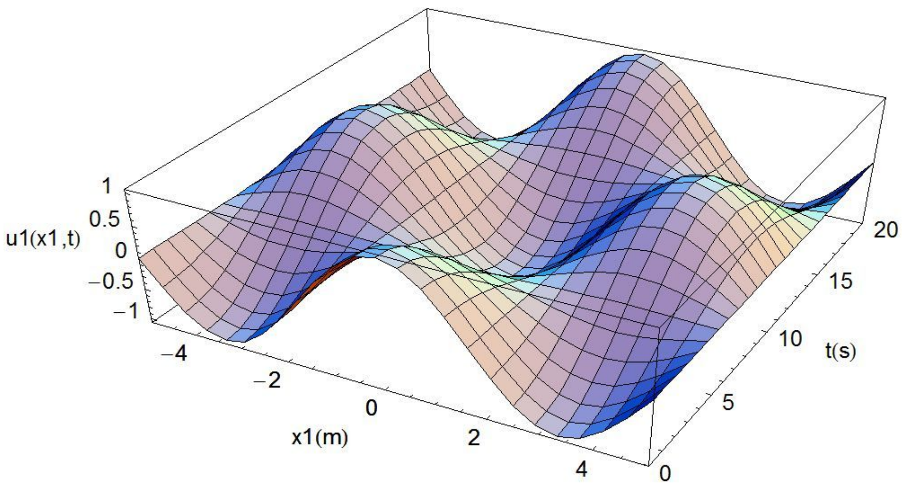

Fig. 3. Diagram of $u_1 = u_1(x_1, t)$, at $n = 1$

Sections by the diagrams of the averaged displacement u_1, u_2 for different times $t = 15s$ is placed on the diagram 4.

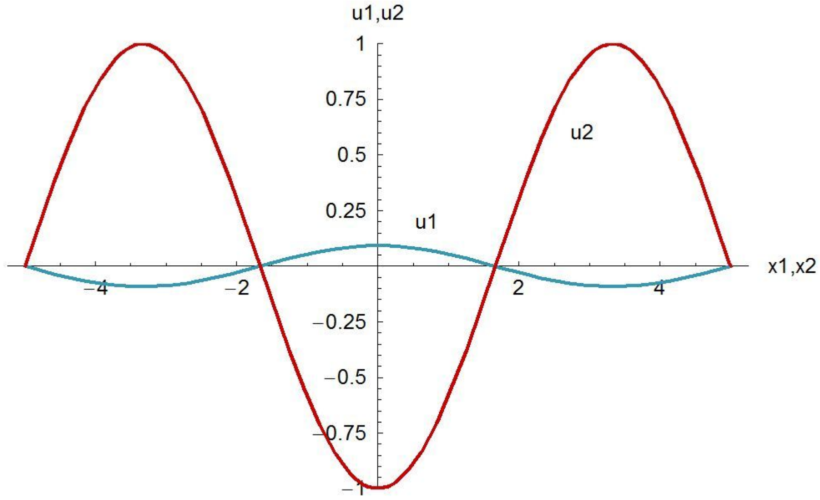

Fig. 4. Diagrams $u_1 = u_1(x_1, t)$, $u_2 = u_2(x_2, t)$, for established time $t = 15s$

Diagrams of the displacement $w_1(x_1, t) = u_1(x_1, t) + h(x_1)v_1(x_1, t)$ and $w_2(x_2, t) = u_2(x_2, t) + h(x_1)v_2(x_2, t)$ are shown on Figures 5, 6.

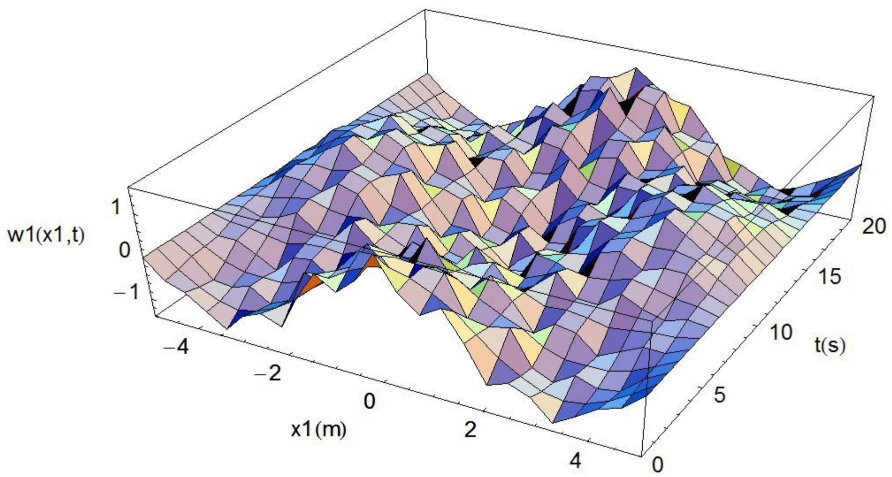

Fig. 5. Diagram of $w_1(x_1, t) = u_1(x_1, t) + h(x_1)v_1(x_1, t)$

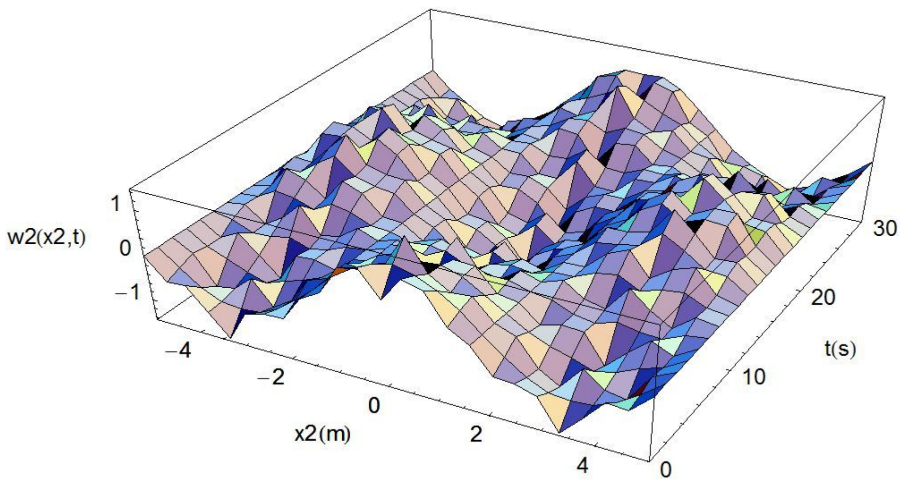

Fig. 6. Diagram of $w_2(x_2, t) = u_2(x_2, t) + h(x_1)v_2(x_2, t)$

Diagram $w_1 = w_1(x_1, t)$ and $w_2 = w_2(x_2, t)$ for established time $t = 30s$, at the free frequency $n = 1$ and at different dimension l (different number of layers), is presented on figures 7, 8.

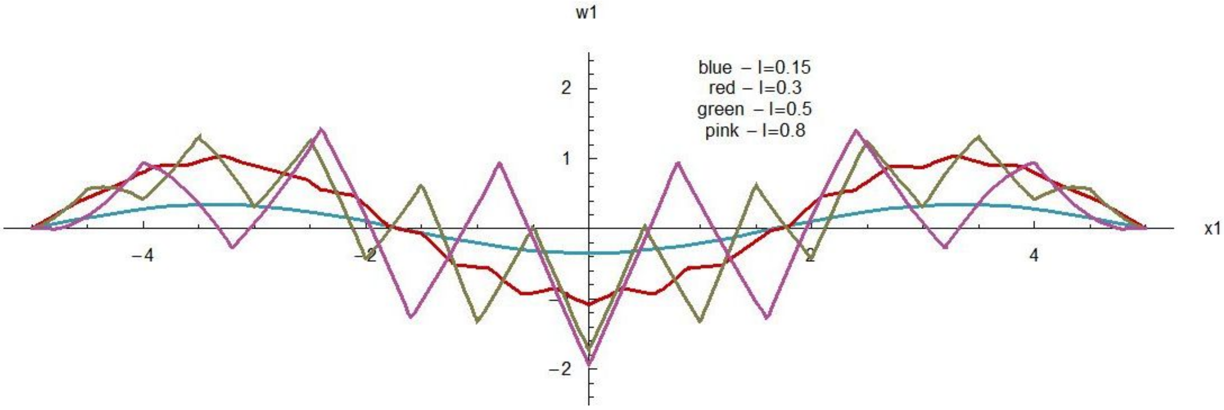

Fig. 7. Diagram $w_1 = w_1(x_1, t)$, for established time $t = 30s$, at different dimension l

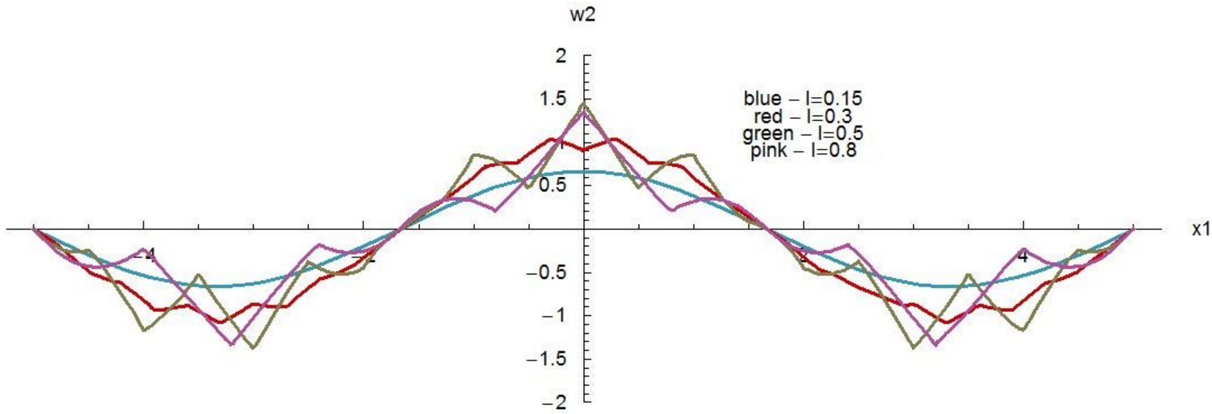

Fig. 8. Diagram $w_2 = w_2(x_2, t)$, for established time $t = 15s$, at different dimension l

Figures 7, 8 clearly show the diversity of the displacement value depending on the basic cell size. So the solutions describe the scale effect .
Differentiation of vibration amplitudes, depending on the plate and ribs material, are presented in figures 9, 10. Considered three options: concrete plate and steel ribs, rubber plate and steel ribs, concrete plate and rubber ribs. Figures 9 and 10 show the diagrams of the displacement $w_1 = w_1(x_1, t)$, $w_2 = w_2(x_2, t)$ for established time $t = 15s$.

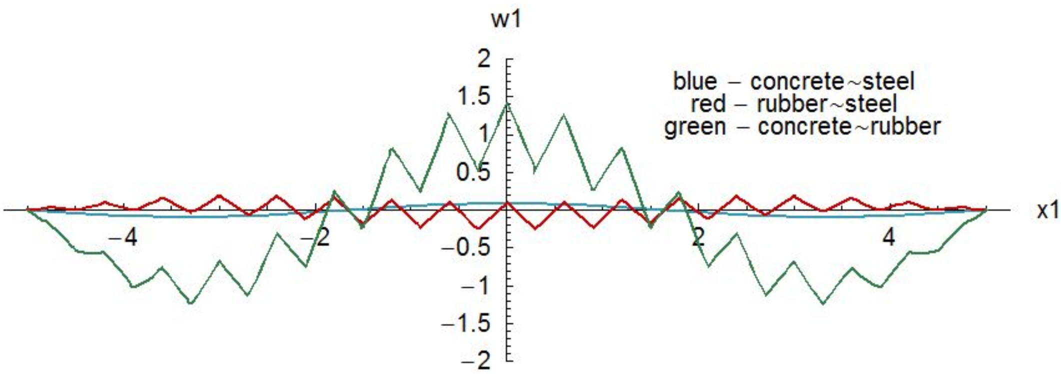

Fig. 9. Section by the diagram $w_1 = w_1(x_1, t)$, for established time $t = 15s$, at different material properties

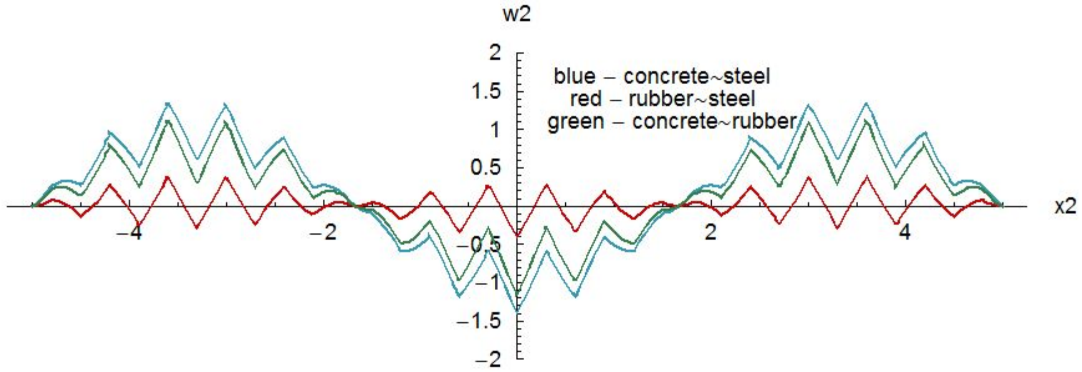

Fig. 10. Section by the diagram $w_2 = w_2(x_2, t)$, for established time t = 15s, at different material properties

CONCLUSIONS

The averaged model of ribbed plates was constructed in this paper. Homogenized technique used in the work did not required a border crossing with the level of basic cell size to zero. So it is possible to explore the scale effect. The clear influence the basic cell dimension on the size of the amplitude of the vibration plate has been shown. It turns out that, in this case the tolerance averaging method is a convenient and effective tool to construct a dynamic model of elastic plates reinforced by periodic arrangement of ribs. The equations in the resulting model are simpler in the comparison to the equations of the elasticity theory. They are in fact differential equations with constant coefficients.

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Katarzyna Jeleniewicz, Wiesław Nagórko
Department of Civil Engineering and Geodesy
Warsaw University of Life Sciences – SGGW
ul. Nowoursynowska 159, 02-776 Warsaw, Poland
e-mail: k.jeleniewicz@tlen.pl

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