Electronic Journal of Polish Agricultural Universities (EJPAU) founded by all Polish Agriculture Universities presents original papers and review articles relevant to all aspects of agricultural sciences. It is target for persons working both in science and industry, regulatory agencies or teaching in agricultural sector. Covered by IFIS Publishing (Food Science and Technology Abstracts), ELSEVIER Science - Food Science and Technology Program, CAS USA (Chemical Abstracts), CABI Publishing UK and ALPSP (Association of Learned and Professional Society Publisher - full membership). Presented in the Master List of Thomson ISI.

ELECTRONIC JOURNAL OF POLISH AGRICULTURAL UNIVERSITIES

2008
Volume 11
Issue 1
Topic
AGRICULTURAL ENGINEERING

STANKIEWICZ A. GOŁACKI K. 2008. APPROXIMATION OF THE CONTINUOUS RELAXATION SPECTRUM OF PLANT VISCOELASTIC MATERIALS USING LAGUERRE FUNCTIONS, EJPAU, 11(1), #20.
Available Online http://www.ejpau.media.pl/volume11/issue1/art-20.pdf

APPROXIMATION OF THE CONTINUOUS RELAXATION SPECTRUM OF PLANT VISCOELASTIC MATERIALS USING LAGUERRE FUNCTIONS

Anna Stankiewicz, Krzysztof Gołacki
Department of Technical Sciences, Agricultural University of Lublin, Poland

ABSTRACT

Relaxation spectrum is a very useful tool in the characterization of viscoelactic materials as the knowledge on them enables the calculation of any linear material functions such as the creep compliance, the Poisson's ratio or shear and bulk modulus. The paper deals with the problem of recovery of continuous relaxation spectrum from discrete-time noise corrupted measurements of relaxation modulus obtained in stress relaxation test. An optimal orthogonal scheme of the least-squares approximation of the spectrum of relaxation frequencies by the finite series of Laguerre functions is presented. The approach proposed allows us to reduce the primary infinite dimensional dynamic optimization problem to a static linear-quadratic programming task. The problem of relaxation spectrum identification is the practical ill-posed problem of reconstructing solution of Fredholm integral equation of the first kind from the measured data. Thus, Tikhonov regularization is used to guarantee the stability of the scheme. Generalized cross validation (GCV) is adopted for the optimal choice of the regularization parameter. The numerical realization of the scheme by using the singular value decomposition (SVD) is discussed and the resulting computer algorithm is outlined. An analysis of the model accuracy is conducted for noise measurements and the linear convergence of the approximations generated by the scheme is proved. It is also indicated that the accuracy of the spectrum approximation depends both on measurement noises and regularization parameter as well as on the proper selection of the basic orthogonal functions. The method combines effectiveness and accuracy and is general enough to cover both viscoelastic solids and liquids. Applying the scheme proposed, the relaxation spectrum of an unconfined cylindrical specimen of the beet sugar root is determined.

Key words: Viscoelasticity, relaxation spectrum, linear relaxation modulus, identification algorithm, regularization, Laguerre functions

INTRODUCTION

Over the last 30 years many advances have been made in the mathematical modelling of the soft biological materials as fruits and vegetables. The need for detailed knowledge of mechanical material functions has been growing with the increased use of accurate engineering methods for rigorous predictions of the plant materials behaviour, such as the finite element method (FEM), the boundary element method (BEM) and the finite difference method [11,13]. Fruits and vegetables are most often modelled in a viscoelastic regime, which is good for characterizing strain-stress dependence, creep and stress relaxation within a small deformation [9,15]. A number of constitutive methods have been developed to describe the time dependent mechanical behaviour of biological materials, e.g. [4,7,9,15,16]. Although for viscoelastic materials, a multiplicity of constitutive theories exists, essentially, only linear viscoelasticity is considered for which the correspondence principle applies. The mechanical properties of linear viscoelastic materials are characterized by relaxation or retardation spectra. From the relaxation spectrum other material functions such as the relaxation modulus or the creep compliance can be calculated without difficulty, and next the time-variable bulk or shear modulus or the time-variable Poisson's ratio can be determined using the identification schemes proposed e.g. in [3,4,16]. The relaxation spectrum can be also used to validate experiments by cross-checking results, e.g., from creep and stress relaxation tests [21]. Thus, the relaxation spectrum is vital not only for constitutive models but also for the insight into the properties of a viscoelastic material.

However, the two spectra are not measurable directly; they must be determined from the appropriate response function, measured either in time or frequency-domain. These calculations require solution of an inverse problem, which happens to be ill-posed. It is one of the classical problems in rheology. Although the literature concerned with the development and investigation of different algorithms for the relaxation spectrum computation using the data from a small-amplitude oscillatory shear experiment is quite extensive, e.g. [2,12,14,21,24] and papers cited therein, there are only few papers, e.g. [4,18,19,20, 22], dealing with the spectrum determination from relaxation modulus data.

The oscillatory-experiment methods are not appropriate for highly hydrated plant materials (fruits and vegetables). A classical manner of studying viscoelasticity for such materials is by two-phase stress relaxation test, where the time-dependent shear stress is studied for step increase in strain [9,15]. Therefore, a computationally efficient methods to determine the relaxation spectrum which can be applied to time-measurements of relaxation modulus obtained in stress relaxation test is desirable and it is the purpose of this study.

In this paper an optimal orthogonal scheme of the least-squares approximation of the spectrum of relaxation frequencies by the linear combination of Laguerre orthonormal functions is proposed. Regularization is introduced for computing the model parameters and implemented by both the singular value decomposition as well as the generalized cross-validation technique.

MATERIALS AND METHODS

Relaxation spectrum

The uniaxial, nonaging and isothermal stress-strain equation for a linear viscoelastic material can be represented by a Boltzmann superposition integral [1]

$$\sigma(t) = \int_{-\infty}^{t} G(t-\lambda)\dot{\varepsilon}(\lambda)\, d\lambda$$

where $\sigma(t)$ and $\varepsilon(t)$ denotes the stress and stain, respectively, and $G(t)$ is the linear relaxation modulus. Modulus $G(t)$ is given by

$$G(t) = \int_{0}^{\infty} H(v)\, e^{-tv} dv \tag{1}$$

where $H(v)$ characterises the distribution of relaxation frequencies in the range $[v, v+dv]$. The relaxation spectrum $H(v)$ is a generalisation of discrete Maxwell spectrum to a continuous function of the relaxation frequencies v.

The problem of relaxation spectrum determination is the practical problem of reconstructing solution of Fredholm integral equation of the first kind (1) of convolution type from time-measured data. This problem is known to be severely Hadamard *ill-posed* [23]. In particular, small changes in measured relaxation modulus can lead to arbitrarily large changes in the relaxation spectrum. In remedy, some reduction of the admissible solutions set or respective regularization of the original problem can be used. We use both the techniques simultaneously. A finite-dimentional approximation of the spectrum by the linear combination of orthogonal Laguerre functions and Tikhonov regularization method will be used.

Model

Assume that $H(v) \in L_2(0,\infty)$, where $L_2(0,\infty)$ is the space of real-valued square-integrable functions on the interval $(0,\infty)$. Let $L_k(v)$, $k \in Z$, the set of integers, be the Laguerre functions defined by the Rodrigues differential formula

$$L_k(v) = \frac{\sqrt{\alpha}}{k!}\, e^{-\alpha v/2}\, \frac{d^k}{d(\alpha v)^k}\left[(\alpha v)^k e^{-\alpha v}\right], \quad k = 0,1\ldots$$

or, equivalently, given by the finite sum of the form

$$L_k(v) = \sqrt{\alpha}\, e^{-\alpha v/2} \sum_{j=0}^{k} \binom{k}{j} \frac{(-\alpha v)^{k-j}}{(k-j)!}, \quad k = 0,1\ldots$$

where a scaling factor $\alpha > 0$. The functions $\{L_k(v)\}$ form an orthonormal basis of the space $L_2(0,\infty)$. The orthogonality property allows us to express the relaxation spectrum as

$$H(v) = \sum_{k=0}^{\infty} g_k\, L_k(v)$$

where g_k are constants. For practical reasons, it is convenient to replace the infinite summation in the above equation with a finite one of K terms

$$H_K(v) = \sum_{k=0}^{K-1} g_k\, L_k(v) \tag{2}$$

Then the respective model of the relaxation modulus is described by

$$G_K(t) = \int_0^{\infty} H_K(v)\, e^{-tv} dv = \sum_{k=0}^{K-1} g_k\, \phi_k(t) \tag{3}$$

where by the well-known Laplace transform of the Laguerre functions we have

$$\phi_k(t) = \int_0^{\infty} L_k(v)\, e^{-tv} dv = (-1)^k \sqrt{\alpha}\, \frac{(t-\alpha/2)^k}{(t+\alpha/2)^{k+1}} = 2(-1)^k \sqrt{\alpha}\, \frac{(2t-\alpha)^k}{(2t+\alpha)^{k+1}}, \quad k = 0,1\ldots \tag{4}$$

Identification problem

Identification consists of selecting within the given class of models defined by (3)-(4) such a model which ensures the best fit to the measurement results. Suppose, a certain identification experiment (stress relaxation test) performed on the specimen of the material under investigation resulted in a set of measurements of the relaxation modulus $\overline{G}(t_i) = G(t_i) + z(t_i)$ at the sampling instants $t_i \geq 0$, $i = 1,\ldots,N$, where $z(t_i)$ is additive measurement noise. As a measure of the model (3)-(4) accuracy the square index is taken

$$Q_N(g_K) = \sum_{i=1}^{N} \left[\overline{G}(t_i) - G_K(t_i)\right]^2 = \left\|\overline{G}_N - \boldsymbol{\Phi}_{N,K}\, g_K\right\|_2^2 \tag{5}$$

where $\|\cdot\|_2$ denotes the quadratic norm in the real euclidean *N*-space R^N (as well as in the space R^K in the sequel), $g_K = [g_0 \quad \ldots \quad g_{K-1}]^T$ is an *K*-element vector of unknown coefficients of the model (3), the matrix $\boldsymbol{\Phi}_{N,K}$ and the vector $\overline{G}_N$ are defined as follows

$$\boldsymbol{\Phi}_{N,K} = \begin{bmatrix} \phi_0(t_1) & \phi_1(t_1) & \ldots & \phi_{K-1}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(t_N) & \phi_1(t_N) & \ldots & \phi_{K-1}(t_N) \end{bmatrix} \qquad \overline{G}_N = \begin{bmatrix} \overline{G}(t_1) \\ \vdots \\ \overline{G}(t_N) \end{bmatrix} \tag{6}$$

Then, the identification problem consists of determining the model parameter g_K minimizing the index (5).

The matrix $\boldsymbol{\Phi}_{N,K}$ is usually ill-conditioned or of deficient rank. Then, the minimum of (5) is not unique and even the normal (minimum norm) solution $\overline{g}_K^N$ of the linear-quadratic problem (5)-(6) is non-continuous and unbounded

function of the data vector $\overline{G}_N$, i.e. when the data are noisy even small changes in $\overline{G}_N$ would lead to arbitrarily large artefact in $\overline{g}_K^N$. Therefore, the numerical solution of finite dimensional problem (5)-(6) is fraught with the same difficulties that the original continuous ill-posed problem. To deal with the ill-posedness, the Tikhonov regularization method is used and presented in the subsequent section.

Regularization

Regularization aims to replace the ill-posed problem by a nearby well-posed problem. Tikhonov regularization [23] strives to stabilize the computation of the minimum norm least-squares solution by minimizing a modified square functional of the form

$$\min_{g_K \in R^K} \left\| \overline{G}_N - \boldsymbol{\Phi}_{N,K}\, g_K \right\|_2^2 + \lambda \left\| g_K \right\|_2^2 \tag{7}$$

where $\lambda > 0$ is a regularization parameter. The above problem is well-posed, that is the solution always exists, is unique, and continuously depends on both the matrix $\boldsymbol{\Phi}_{N,K}$ as well as on the measurement data $\overline{G}_N$. Minimizing (7) the optimal vector is given by

$$g_K^{\lambda} = \left(\boldsymbol{\Phi}_{N,K}^T\, \boldsymbol{\Phi}_{N,K} + \lambda\, I_{K,K} \right)^{-1} \boldsymbol{\Phi}_{N,K}^T\, \overline{G}_N \tag{8}$$

where superscript "*T*" indicates transpose and $I_{K,K}$ is $K \times K$ identity matrix.

The choice of regularization parameter λ is crucial to identify the best model parameters. There exist different ways of suitable choosing the regularization parameter [2, 8]. They differ in the amount of *a priori* information required as well as in the decision criteria. Here we employ the generalized cross-validation of Golub, Heath and Wahba [6], which does not depend on *a priori* knowledge about the noise variance. The GCV technique relies on choosing as regularization parameter that $\hat{\lambda}$ which minimizes the GCV functional defined by

$$V(\lambda) = \left\| r(\lambda) \right\|_2^2 \Big/ tr\left[M(\lambda) \right]^2 \tag{9}$$

where

$$M(\lambda) = I_{N,N} - \boldsymbol{\Phi}_{N,K} \left(\boldsymbol{\Phi}_{N,K}^T\, \boldsymbol{\Phi}_{N,K} + \lambda\, I_{K,K} \right)^{-1} \boldsymbol{\Phi}_{N,K}^T$$

and $r(\lambda) = M(\lambda)\overline{G}_N = \overline{G}_N - \boldsymbol{\Phi}_{N,K}\, g_K^{\lambda}$ is the residual vector for the regularized solution (8), $tr\left[M(\lambda) \right]$ denotes the trace of $M(\lambda)$.

Algebraic background

For the computational purposes the elegant formula (8) is generally unsuitable. For numerical computation of regularized solution, the singular value decomposition technique will be used.

Let SVD of the matrix $\boldsymbol{\Phi}_{N,K}$ take the form [5]

$$\boldsymbol{\Phi}_{N,K} = U\, \boldsymbol{\Sigma}\, V^T \tag{10}$$

where V is orthogonal matrix (i.e., the columns of V are orthonormal) of order $K \times K$, U is $N \times N$ orthogonal matrix and $\boldsymbol{\Sigma} = diag(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$ is $N \times K$ diagonal matrix containing the non-zero singular values $\sigma_1, \ldots, \sigma_r$ of the matrix $\boldsymbol{\Phi}_{N,K}$ with $r = rank\left(\boldsymbol{\Phi}_{N,K} \right)$. Taking advantage of the diagonal structure of $\boldsymbol{\Sigma}^T \boldsymbol{\Sigma}$ and the matrices V and U orthogonality, it may be simply proved that

$$g_K^{\lambda} = V\, \boldsymbol{\Omega}_{\lambda}^{-1}\, V^T\, \boldsymbol{\Phi}_{N,K}^T\, \overline{G}_N \tag{11}$$

where $\boldsymbol{\Omega}_{\lambda} = \left(\boldsymbol{\Sigma}^T \boldsymbol{\Sigma} + \lambda\, I_{K,K} \right)$, and whence

$$\boldsymbol{\Omega}_{\lambda}^{-1} = diag\left(1/\left(\sigma_1^2 + \lambda \right), \ldots, 1/\left(\sigma_r^2 + \lambda \right), 1/\lambda, \ldots, 1/\lambda \right) \tag{12}$$

Using SVD (10) of $\boldsymbol{\Phi}_{N,K}$ the GCV function can be expressed by a convenient analytical formula [19]

$$V(\lambda)=\left[\sum_{i=1}^{r}\frac{\lambda^2 y_i^2}{(\sigma_i^2+\lambda)^2}+\sum_{i=r+1}^{N}y_i^2\right]\Bigg/\left[N-r+\sum_{i=1}^{r}\frac{\lambda}{\sigma_i^2+\lambda}\right]^2 \tag{13}$$

as a function of the singular values $\sigma_1,\dots,\sigma_r$ and the elements of the wektor $Y=U^T\ \overline{G}_N$. It is easy to check that $dV(0)/d\lambda<0$, and whence the optimal parameter $\hat{\lambda}>0$. The GCV function (13) is differentiable for any λ, thus an arbitrary gradient optimization method can be implemented to solve the GCV minimization task.

RESULTS AND DISCUSSION

Analysis

The purpose of the regularization relies on stabilization of the resulting vector g_K^{λ}. The effectiveness of this approach can be evaluated by the following relations which follow immediately from Proposition 1 in [18]

$$\left\|g_K^{\lambda}\right\|_2^2=\sum_{i=1}^{r}\frac{\sigma_i^2\ y_i^2}{(\sigma_i^2+\lambda)^2}<\sum_{i=1}^{r}\frac{y_i^2}{\sigma_i^2}=\left\|\overline{g}_K^N\right\|_2^2\le\left(\left\|g_K^N\right\|_2+\frac{1}{\sigma_r}\left\|z_N\right\|_2\right)^2 \tag{14}$$

where $z_N=[z(t_1)\ \dots\ z(t_N)]^T$ and g_K^N is the normal solution of the linear-quadratic task (5)-(6) for noise-free measurement data. The first equality in (14) illustrates the mechanism of stabilization. The following rule holds: the greater the regularization parameter λ is, the fluctuations of the vector g_K^{λ} are highly bounded. Thus, the regularization parameter controls the "smoothness" of the regularized solution.

By orthonormality of the basic functions $\{L_k(v)\}$ in the space $L_2(0,\infty)$, for an arbitrary $H_K(v)$ of the form (2) the following equality holds

$$\left\|H_K\right\|_2^2=\sum_{k=0}^{K-1}\sum_{j=0}^{K-1}g_k g_j\int_0^{\infty}L_k(v)L_j(v)dv=\sum_{k=0}^{K-1}g_k{}^2=\left\|g_K\right\|_2^2 \tag{15}$$

where $\|\cdot\|_2$ means the (square) norm in $L_2(0,\infty)$. Therefore, the smoothness of the discrete problem (7) optimal solution guarantees that the fluctuations of the respective spectrum of relaxation, in particular the resulting spectrum of relaxation

$$\hat{H}_K(v)=\sum_{k=0}^{K-1}g_K^{\hat{\lambda}}\,L_k(v) \tag{16}$$

are also bounded. In view of the above, the function $\hat{H}_K(v)$ (16) is the approximation of the real spectrum $H(v)$ in the class of functions (2) optimal in the sense of the square identification index (5) of the bounded norm $\left\|\hat{H}_K\right\|_1<\left\|\overline{g}_K^N\right\|$.

Relaxation spectrum $\hat{H}_K(v)$ (16) is only approximation of that spectrum, which can be obtained in the chosen class of models (3)-(4) by direct minimization (without regularization) of the quality index (5) for noise-free measurements of the relaxation modulus, i.e. the approximation of the function $H_K^N(v)=\sum_{k=0}^{K-1}g_k^N\,L_k(v)$. Taking into account the equality (15) the following bound of the approximation error can be derived (see Appendix)

$$\left\|\hat{H}_K-H_K^N\right\|_2=\left\|g_K^{\hat{\lambda}}-g_K^N\right\|_2\le\hat{\lambda}\sum_{i=1}^{r}\frac{|y_i|}{\sigma_i^3}+\sum_{i=1}^{r}\frac{1}{\sigma_i}\left\|z_N\right\|_2 \tag{17}$$

Therefore, the regularized vector $g_K^{\hat{\lambda}}$ converges to the normal solution g_K^N, linearly with respect to the norm $\|z_N\|_2$, as $\hat{\lambda}\to 0$ and $\|z_N\|_2\to 0$, simultaneously. Likewise, the inequality in (17) guarantees that the spectrum $\hat{H}_K(v)$ tends to $H_K^N(v)$ in each point v, at which they are both continuous, as $\hat{\lambda}\to 0$ and $\|z_N\|_2\to 0$, simultaneously.

The inequality (17) yield that the accuracy of the spectrum approximation depends both on the measurement noises and the regularization parameter as well as on the singular values $\sigma_1,\dots,\sigma_r$ of the matrix (6), which, in turn, depend on the proper selection of the basic orthogonal functions $L_0(v),\dots,L_{K-1}(v)$.

Identification scheme

Allowing the above, the calculation of the relaxation model involves the following steps.

1. Perform the experiment – stress relaxation test [7,15] – and record the measurements $\overline{G}(t_i)$, $i = 1, \ldots, N$, of the relaxation modulus at times $t_i \geq 0$.
2. Compute the square matrix $\boldsymbol{\Phi}_{N,K}$ (6), and next determine SVD (10) of $\boldsymbol{\Phi}_{N,K}$.
3. Determine GCV function $V(\lambda)$ (13), and next compute the optimal regularization parameter $\hat{\lambda} = arg\ min\ V(\lambda)$.
4. Determine $K \times K$ – order inverse matrix (12) for $\lambda = \hat{\lambda}$.
5. Compute the regularized solution $g_K^{\hat{\lambda}}$ according to the formula (11) for $\lambda = \hat{\lambda}$.
6. Determine the spectrum of relaxation frequencies $\hat{H}_K(\nu)$ as a linear combination of Laguerre functions (16).

Remark 1: Only the SVD of the matrix $\boldsymbol{\Phi}_{N,K}$ is space and time consuming task of the scheme. Efficient serial algorithms for SVD are available nowadays, based on the method developed by Golub and Reinsch [5]. However, the operations to perform SVD on a $N \times K$ matrix have the time complexity $O(NK^2)$, therefore of late block-based SVD are developed, e.g. [10]. Note however, that in the scheme proposed, the SVD must be computed only once.

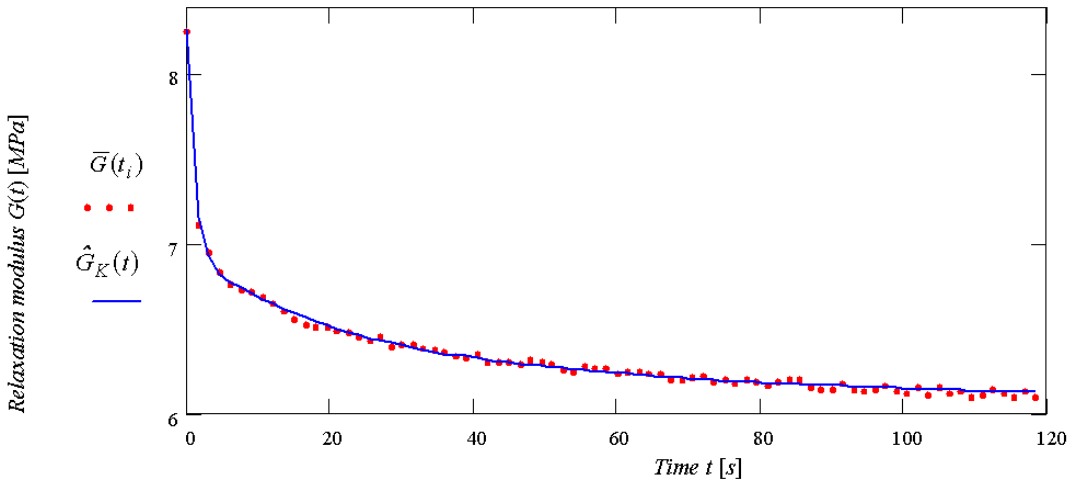
Remark 2: For the scaling factor in the Laguerre functions α the following rule holds: the low the parameter α is, the shorter the relaxation times are, i.e. the greater are the relaxation frequencies.

Remark 3: By the optimal choice of the scaling factor, the best fit of the model to the experimental data can be achieved. In such a case, a two-level hierarchical identification scheme – compare [17] – can be used, in which on the first lower level for a given value of α the above procedure is realized and on the second upper level the optimal parameter α is computed by minimizing the original model quality index $Q_N(g_K)$. However, in practice a simple rough rule for choosing the scaling factor α, based on the comparison of a few first functions from the sequence $\{\phi_k(t)\}$ for different values of α with the experimentally obtained function $\overline{G}(t)$ is quite enough. In the same manner, the number K of the series (3) elements can be initially evaluated. Thus, the choice both of the number K as well as the parameter α must be done *a posteriori*, after the preliminary experiment data analysis.

Example

A cylindrical sample of 20 mm diameter and height was obtained from the root of sugar beet Oktawia variety. The sample was tested on the universal testing machine INSTRON 6022. During the two-phase stress relaxation test, in the first initial phase the strain was imposed instantaneously, the sample was preconditioned at the 0.833 mm·s^{-1} strain rate to the maximum strain. Next, during the second phase at constant strain the corresponding force induced in the specimen, which decreases with time, was recorded during the time period [0,120] seconds. The experiment was performed in the state of uniaxial stress, i.e. the specimen examined underwent deformation between two parallel plates. The respective relaxation modulus was computed as the ratio of the stress to the strain imposed and plotted in Fig. 1. Next the proposed identification scheme was applied.

Fig. 1. Relaxation modulus $\overline{G}(t_i)$ corrupted by additive noise (points) and the optimal mathematical model $\hat{G}_K(t)$ (solid line)



Since the relaxation modulus was recorded only over 120 s a constant value known as the long-term modulus G_E is introduces in the classical integral model (1), i.e. the modified model of the form

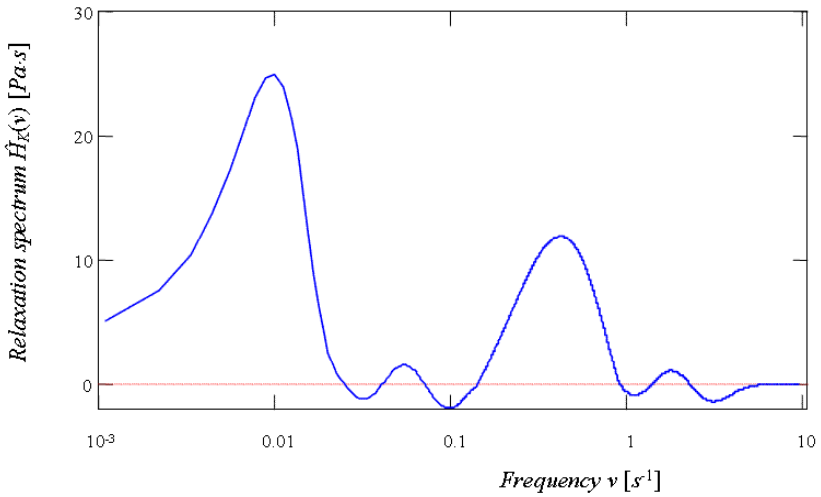
$$G(t) = \int_0^\infty H(v)\, e^{-tv} dv + G_E$$

is used to guarantee the better fit of the experiment data. It is easy to check that for the above model the matrix $\boldsymbol{\Phi}_{N,K}$ and the vector g_K must be defined as

$$\boldsymbol{\Phi}_{N,K} = \begin{bmatrix} \phi_0(t_1) & \dots & \phi_{K-1}(t_1) & 1 \\ \vdots & \ddots & \vdots & \vdots \\ \phi_0(t_N) & \dots & \phi_{K-1}(t_N) & 1 \end{bmatrix} \qquad g_K = [g_0 \quad \dots \quad g_{K-1} \quad G_E]^T$$

The optimal regularization parameter is $\hat{\lambda} = 4.6E\text{-}5$. The norm of the regularized vector, i.e. according to equation (15), the norm of the "smoothed" spectrum is equal $\left\|\hat{H}_K\right\|_1 = 9.697$ and the constant $G_E = 5.994$ MPa. The optimal identification index is $\hat{Q}_N = 0.0885$, thus the model (3)-(4) fitted the data extremely well, as indicated especially by the mean-square model error $\hat{Q}_N / N = 0.000369$. The optimal model of the relaxation modulus given, taking account of (3), by $\hat{G}_K(t) = \sum_{k=0}^{K-1} g_k^{\hat{\lambda}}\, \phi_k(t)$ is plotted in Fig. 1 and the resulting relaxation spectrum is plotted in Fig. 2.

Fig. 2. Relaxation spectrum $\hat{H}_K(v)$ versus frequency v



CONCLUSIONS

A robust algorithm has been found for the calculation of relaxation frequencies from the measurement data of the linear relaxation modulus discrete-time measurements. The approach proposed is based on the approximation of the spectrum by finite linear combination of the basic Laguerre functions. As a result, the primary infinite dimensional dynamic optimization problem of the continuous relaxation spectrum identification is reduced to the static linear-quadratic programming task. Tihonov regularization and generalized cross validation are used to solve it, thus the stability of the resulting scheme is guaranteed.

The choice of the orthogonal basic functions guarantees that smoothing of the regularized solution of the linear-quadratic problem ensures smoothing of the resulting relaxation spectrum model. Due to the choice of the Laguerre basic functions, for which the integrals (4) are given by the simple analytical formula, the errors of the approximate quadrature rules required in other known methods, see e.g. [14], are avoided. The above is of great importance in the context of ill-posed inverse problem.

An analysis of the model accuracy is conducted both in the case of perfect (noise-free) and corrupted by additive noise measurements of the linear relaxation modulus and the linear convergence of the approximations generated by the scheme is proved. The choice of the scaling factor in order to achieve a good fit of the model to the experiment data is discussed. The numerical experimental studies suggest that the scheme proposed can be successfully used within a satisfactory range of both viscoelastic solids and liquids.

APPENDIX

It is well-known that the normal solution of the linear-quadratic task (5)-(6) $\overline{g}_K^N = \boldsymbol{\Phi}_{N,K}^{\dagger}\,\overline{G}_N$ and that $\boldsymbol{\Phi}_{N,K}^{\dagger}$, the Moore-Penrose inverse of $\boldsymbol{\Phi}_{N,K}$, can be expressed using SVD (10) as $\boldsymbol{\Phi}_{N,K}^{\dagger} = V\,\Sigma^{\dagger}\,U^T$, where $\Sigma^{\dagger} = diag\left(1/\sigma_1,\ldots,1/\sigma_r,0,\ldots,0\right)\in R^{K,N}$. Since on the basis of (10) and (11) the regularized vector $g_K^{\hat{\lambda}} = V\,\Omega_{\hat{\lambda}}^{-1}\Sigma^T\,U^T\overline{G}_N$ the approximation error $g_K^{\hat{\lambda}} - \overline{g}_K^N$ can be rewritten as $g_K^{\hat{\lambda}} - \overline{g}_K^N = V\left(\Omega_{\hat{\lambda}}^{-1}\Sigma^T - \Sigma^{\dagger}\right)Y$, where $Y = U^T\overline{G}_N$. Hence, by virtue of the diagonal structure of the matrices $\Omega_{\hat{\lambda}}^{-1}$ (12), Σ and $\Sigma^{\dagger}$ we obtain

$$\left\|g_K^{\hat{\lambda}} - \overline{g}_K^N\right\|_2^2 = Y^T\left(\Omega_{\hat{\lambda}}^{-1}\Sigma^T - \Sigma^{\dagger}\right)^T\left(\Omega_{\hat{\lambda}}^{-1}\Sigma^T - \Sigma^{\dagger}\right)Y = \sum_{i=1}^{r}\frac{\hat{\lambda}^2 y_i^2}{\sigma_i^2\left(\sigma_i^2+\hat{\lambda}\right)^2}$$

which imply the following sequence of bounds

$$\left\|g_K^{\hat{\lambda}} - \overline{g}_K^N\right\|_2^2 \le \left[\sum_{i=1}^{r}\frac{\hat{\lambda}\,|y_i|}{\sigma_i\left(\sigma_i^2+\hat{\lambda}\right)}\right]^2 \le \left[\sum_{i=1}^{r}\frac{\hat{\lambda}\,|y_i|}{\sigma_i^3}\right]^2 \tag{A1}$$

Similarly, since $\overline{g}_K^N - g_K^N = V\,\Sigma^{\dagger}\,U^T\overline{G}_N - V\,\Sigma^{\dagger}\,U^T G_N = V\,\Sigma^{\dagger}\,U^T z_N$, we find that

$$\left\|\overline{g}_K^N - \overline{g}_K^N\right\|_2^2 = z^T U\,\Sigma^{\dagger T}\Sigma^{\dagger}\,U^T z = \sum_{i=1}^{r}\frac{\left(u_i^T z\right)^2}{\sigma_i^2}$$

where u_i is the i-th column of the matrix U. Hence, we obtain

$$\left\|\overline{g}_K^N - \overline{g}_K^N\right\|_2^2 \le \left[\sum_{i=1}^{r}\frac{\left|u_i^T z\right|}{\sigma_i}\right]^2 \le \left[\sum_{i=1}^{r}\frac{1}{\sigma_i}\right]^2\|z\|_2^2 \tag{A2}$$

since, according to the Schwarz inequality using the orthonormality of the vectors u_i, we have $\left|u_i^T z\right| \le \|u_i\|_2\|z\|_2 = \|z\|_2$. Now, the estimation (17) follows immediately from (A1), (A2) and the triangle inequality $\left\|g_K^{\hat{\lambda}} - g_K^N\right\|_2 \le \left\|g_K^{\hat{\lambda}} - \overline{g}_K^N\right\|_2 + \left\|\overline{g}_K^N - g_K^N\right\|_2$.

REFERENCES

1. Christensen R.M., 1971. Theory of Viscoelasticity. An Introduction. Academic Press, New York.
2. Elster C., Honerkamp J., Weese J., 1991. Using regularization methods for the determination of relaxation and retardation spectra of polymeric liquids. Rheological Acta, 30(2), 161-174.
3. Emri I., von Bernstorff B. S., Cvelbar R., Nikonov A., 2005. Re-examination of the approximate methods for interconversion between frequency- and time-dependent material functions. J. Non-Newtonian Fluid Mech. 129, 75–84.
4. Fujihara S., Yamamoto R., Masuda Y., 1995. Maxwellian spectra of stress relaxation in the cell wall and growth regulation in higher plants. In: Proceedings of International Workshop Stress Relaxation in Solids and Biological Origin, 47-51. J. Blachovec, ed., Czech Univ. of Agriculture Publisher, Prague.
5. Golub G., van Loan C., 1996. Matrix Computations. Johns Hopkins University Press, Baltimore.
6. Golub G.H., Heath M.T., Wahba G., 1979. Generalized cross-validation as a method for choosing a good ridge parameter. Technometrics 21(2), 215–223.
7. Gołacki K., 1995. Some applications of the results of carrot roots stress relaxation test. In: Proceedings of International Workshop Stress Relaxation in Solids and Biological Origin, pp. 65-69. J. Blachovec, ed., Czech Univ. of Agriculture Publisher, Prague.
8. Hansen P.C., 1997. Rank-Deficient and Discrete Ill-Posed Problems – Numerical Aspects of Linear Inversion. SIAM, Philadelphia.
9. Hellebrand H.J., 1995. Comparison of models for evaluation of stress relaxation. In: Proceedings of International Workshop Stress Relaxation in Solids and Biological Origin, 3-10. J. Blachovec, ed., Czech Univ. of Agriculture Publisher, Prague.
10. Konstantinides K., Natarajan B., Yovanof G.S., 1997. Noise estimation and filtering using block-based singular value decomposition. IEEE Transact. Image Proc. 6(3), 479-483.

11. Lemaitre J. (ed.), 2001. Handbook of Materials Behavior Models. Academic Press, New York.
12. Malkin A.Ya., Masalova I., 2001. From dynamic modulus via different relaxation spectra to relaxation and creep functions. Rheological Acta 40(3), 261-271.
13. Owens R.G., Phillips T.N., 2002. Computational Rheology. Imperial College Press, London.
14. Paulson K.S., Jouravleva S., McLeod C.N., 2000. Dielectric relaxation time spectroscopy. IEEE Transact. Biomed. Eng. 47(11), 1510-1517.
15. Rao M.A., 1999. Rheology of Fluid and Semisolid Foods. Principles and Applications. Aspen Publishers, Inc., Gaithersburg, Maryland.
16. Stankiewicz A., Gołacki K., 2004. Algorytm identyfikacji zmiennych w czasie modułów odkształcenia postaciowego i objętościowego roślinnych materiałów lepkosprężystych [A scheme for identification of time-variable shear and bulk modulus of plant viscoelastic materials]. Acta Sci. Pol., Technica Agraria 3(1-2), 79-95 [in Polish].
17. Stankiewicz A., 1995. A decentralized three-level scheme for global identification of complex steady-state systems. In: Proceedings of the Second International Symposium on Methods and Models in Automation and Robotics, 299-304. S. Banka, ed. Technical University of Szczecin Publisher, Szczecin.
18. Stankiewicz A., 2003. Algorytm identyfikacji ciągłego spektrum czasów relaksacji biologicznych materiałów lepkosprężystych [A scheme for identification of continuous relaxation time spectrum of viscoelastic plant materials]. Acta Sci. Pol., Technica Agraria 2(2), 77-91 [in Polish].
19. Stankiewicz A., 2005. Algorytm aproksymacji spektrum retardacji roślinnych materiałów lepkosprężystych funkcjami Legendre'a [Approximation of the retardation spectrum of plant viscoelastic materials using Legendre functions]. Acta Agrophysica 6(3), 795-813 [in Polish].
20. Stankiewicz A., 2007. A scheme for direct approximation of continuous relaxation spectrum of viscoelastic plant materials. Ann. Rev. Agric. Eng. [in press].
21. Syed Mustapha S.M.F.D., Phillips T.N., 2000. A dynamic nonlinear regression method for the determination of the discrete relaxation spectrum. J. Physics D: Applied Physics 33(10), 1219-1229.
22. Ter Haar D., 1950. A phenomenological theory of visco-elastic behaviour. Physica, XVI, 719-737.
23. Tikhonov A.N., Arsenin V.Y., 1977. Solutions of Ill-posed Problems. John Wiley and Sons New York, USA.
24. Winter H.H., 1997. Analysis of dynamical mechanical data: inversion into a relaxation time spectrum and consistency check. J. Non-Newtonian Fluid Mech. 68(2-3), 225-239.

FOOTNOTE

A preliminary version of the paper was presented in 2nd Technical Symposium of CIGR Section VI, CIGR 2006 Future of Food Engineering, 26-28 April 2006, Warsaw, Poland.

Anna Stankiewicz, Krzysztof Gołacki
Department of Technical Sciences
Agricultural University of Lublin, Poland
50A Doświadczalna Street, 20-280 Lublin, Poland
phone: (+48 81) 4610061 ext. 253
e-mail: anna.stankiewicz@ar.lublin.pl, krzysztof.golacki@ar.lublin.pl

---
